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On Generalized Continuous Fuzzy Proper Function from a Fuzzy Topological Space to another Fuzzy Topological Space

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Abstract:

The purpose of this paper is to introduce and study the concepts of fuzzy generalized open sets, fuzzy generalized closed sets, generalized continuous fuzzy proper functions and prove results about these concepts.

Key words: Fuzzy generalized closed sets, Fuzzy proper function, Generalized continuous fuzzy proper functions.

Introduction and preliminaries:

The concept of fuzzy set and fuzzy set operations were first introduced by Zadeh's (1) .The notation of a fuzzy subsets naturally plays a significant role in the study of fuzzy topology was introduced by Chang (2).

The fuzzy α −open sets and fuzzy α −continuous mapping were introduced and generalized by Bin Shahna (3).

Balasubramanian and Sundaram (4) defined fuzzy generalized closed set in fuzzy topological space (or f.t.s. in short) on X. I^X is denoted the collection of all mapping from X in to I=[0,1]. A fuzzy set $\tilde{A}$ in f.t.s $(X,\tilde{T})$ is said to be quasicoincident (or q-coincident in short) with a fuzzy set $\tilde{B}$, denoted by $\tilde{A}q\,\tilde{B}$, if there exists $x \in X$ such that $\mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) > 1$ (5). If $\tilde{A},\tilde{B} \in I^x$, $\mu_{\tilde{B}}(x) \le \mu_{\tilde{A}}(x)\ \forall\, x \in X$, then $\tilde{B}$ is said to be a fuzzy subset of $\tilde{A}$ and denoted by $\tilde{B} \subseteq \tilde{A}$ (2). A fuzzy point $P^r_{x_\circ}$ in a set X is a fuzzy set with membership function $\mu_{P^r_{x_\circ}}(x)$, defined by:

$$\mu_{P^r_{x_\circ}}(x) = r \text{ for } x = x_P,$$

and $$\mu_{P^r_{x_\circ}}(x) = 0 \text{ for } x \ne x_P.$$

where $r \in (0,1]$, $x_\circ$ is called the support of $P^r_{x_\circ}$ and r the value of $P^r_{x_\circ}$ (6). A fuzzy point $P^r_{x_\circ}$ in X is called belong to a fuzzy set $\tilde{A}$ in X (notation: $P^r_{x_\circ} \in \tilde{A}$) iff

$r \le \mu_{\hat{A}}(x)$, a fuzzy set $\tilde{A}$ in X is the union of all its fuzzy points (7).

In the present paper we study the properties of generalized continuous fuzzy proper functions from a fuzzy topological space on a fuzzy set $\tilde{A}$ to a fuzzy topological space on a fuzzy set $\tilde{B}$.

1. Basic Concept of a Fuzzy Topological Space on Fuzzy Set $\tilde{A}$

In this section, we present fuzzy topological space on fuzzy set with fundamental concepts in fuzzy topological space on fuzzy set, such as quasi-coincident, complement of fuzzy set, maximal fuzzy set, etc.

Remark (1.1), (8):

Let $\tilde{A} \in I^X, \tilde{C} \in I^Y, P(\tilde{A}) = \{\tilde{B} \in I^X : \tilde{B} \subseteq \tilde{A}\}$ and $P(\tilde{C}) = \{\tilde{D} \in I^Y : \tilde{D} \subseteq \tilde{C}\}$.

Definition (1.1), (8):

Let $\tilde{A},\tilde{B}$, be two fuzzy set in X with $\tilde{B} \in P(\tilde{A})$ then the complement of $\tilde{B}$ reletive to $\tilde{A}$, denoted by $(\tilde{B})^c$, is defined by:

$\mu_{(\tilde{B})^c}(x) = \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)$.

Definition (1.2)(7):

Let $P^r_x \in \tilde{A}$ and $\tilde{B} \in P(\tilde{A})$ are said to be q-coincident relative to $\tilde{A}$ [written as $P^r_x q_{\tilde{A}} \tilde{B}$] if there exists $x \in X$, such that $r + \mu_{\tilde{B}}(x) > \mu_{\tilde{A}}(x)$. If P^r_x is not q-coincident with $\tilde{B}$ in $\tilde{A}$, we denoted for this . $P^r_x \bar{q}_A \tilde{B}$.

Definition (1.3), (9):

If $\tilde{B} \in P(\tilde{A})$ then $\tilde{B}$ is said to be maximal if $\forall\, x \in X$, $\mu_{\hat{B}}(x) \ne 0$, then

$\mu_{\hat{B}}(x) = \mu_{\hat{A}}(x)$.

Definition (1.4)(10):

A collection $\tilde{T}$ of a fuzzy subsets of $\tilde{A}$ is said to be fuzzy topology on $\tilde{A}$, if :

(a) $\tilde{\emptyset}, \tilde{A} \in \tilde{T}$.

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(b) If $\tilde{B}, \tilde{C} \in \tilde{T}$, then $\tilde{B} \cap \tilde{C} \in \tilde{T}$.
(c) If $\tilde{B}_j \in \tilde{T}$, $\forall$ j $\in$ J, where J is any index set, then $\bigcup_{j\in J} \tilde{B}_j \in \tilde{T}$.

$(\tilde{A}, \tilde{T})$ is said to be a fuzzy topological space on fuzzy set $\tilde{A}$ (or f.t.s on $\tilde{A}$) and the members of $\tilde{T}$ are said to be fuzzy open sets of $\tilde{A}$. We denote $\hat{T}^c$ the family of fuzzy closed sets of $\tilde{A}$, that is $\tilde{B} \in \tilde{T}^c$ if and only if $\mu_{\tilde{B}^c}(x) = \mu_{\hat{A}}(x) - \mu_{\tilde{B}}(x) \in \tilde{T}$.

Definition (1.5), (10):

Let $\tilde{B}$ be a fuzzy set in a f.t.s $(\tilde{A}, \tilde{T})$. The closure $\bar{\tilde{B}}$ (or cl($\tilde{B}$)) and interior $\tilde{B}^\circ$ (or int($\tilde{B}$)) of $\tilde{B}$ are defined, respectively, by:

cl($\tilde{B}$) = $\cap \{\tilde{F} : \tilde{B} \subseteq \tilde{F}, \tilde{F}^c \in \tilde{T}\}$

int($\tilde{B}$)= $\cup \{\tilde{G} : \tilde{G} \subseteq \tilde{B}, \tilde{G} \in \tilde{T}\}$

2.Fuzzy Generalized Open Set and Fuzzy Generalized Closed Set in a Fuzzy Topological Space on fuzzy set

In this section, we give definitions fuzzy generalized closed sets, fuzzy generalized open sets and fuzzy proper function with some properties.

Definition (2.1):

A fuzzy set $\tilde{B}$ in a fuzzy topological space $(\tilde{A}, \tilde{T})$ is said to be

1) fuzzy generalized closed set (in short, f.g.c.s.) if cl($\tilde{B}$) $\subseteq \tilde{G}$, whenever $\tilde{B} \subseteq \tilde{G}$ and $\tilde{G}$ is fuzzy open set in $(\tilde{A}, \tilde{T})$ (11).
2) fuzzy generalized open set (in short, f.g.o.s) if $\tilde{F} \subseteq$ int($\tilde{B}$), whenever $\tilde{F} \subseteq \tilde{B}$ and $\tilde{F}$ is fuzzy closed set in $(\tilde{A}, \tilde{T})$(12).

Theorem (2.1)(12):

In a fuzzy topological space $(\tilde{A}, \tilde{T})$ the complement of a fuzzy generalized closed set is fuzzy generalized open set.

Theorem (2.2)(12):

Let $(\tilde{A}, \tilde{T})$ be a fuzzy topological space

a. If $\tilde{B}_1$and $\tilde{B}_2$ are f.g.o.s.in $(\tilde{A}, \tilde{T})$, then $\tilde{B}_1 \cap \tilde{B}_2$ is f.g.o.s. in $(\tilde{A}, \tilde{T})$.
b. If $\tilde{B}_1$and $\tilde{B}_2$ are f.g.c.s. in $(\tilde{A}, \tilde{T})$, then $\tilde{B}_1 \cup \tilde{B}_2$ is f.g.c.s. in $(\tilde{A}, \tilde{T})$.

Remark (2.1)(12):

Let $(\tilde{A}, \tilde{T})$ a fuzzy topological space. in general The intersection of two f.g.c.s. in $(\tilde{A}, \tilde{T})$. is not generally a f.g.c.s. in $(\tilde{A}, \tilde{T})$ and
the union of two f.g.o.s. in $(\tilde{A}, \tilde{T})$.is not generally a f.g.o.s. in $(\tilde{A}, \tilde{T})$ as the following example

Example (2.1):

Let X = {a, b} and fuzzy subsets of X are defined as follows:

$\tilde{A}$ = {(a, 0.7), (b, 0.6)}, $\tilde{B}$ = {(a, 0.4), (b, 0.4)}
$\tilde{C}$ = {(a, 0.5), (b, 0.2)}, $\tilde{D}$ = {(a, 0.4), (b, 0.5)},
$\tilde{E}$ = {(a, 0.3), (b, 0.1)}, $\tilde{F}$ = {(a, 0.2), (b, 0.4)}

The collection $\tilde{T} = \{\hat{\emptyset}, \tilde{A}, \tilde{B}\}$ is a fuzzy topology on $\tilde{A}$. The fuzzy sets $\tilde{C}$ and $\tilde{D}$ are f.g.c.s but $\tilde{C} \cap \tilde{D}$ is not a f.g.c.s. in $\tilde{A}$, and the fuzzy sets $\tilde{E}$ and $\tilde{F}$ are f.g.o.s in $\tilde{A}$, but $\tilde{E} \cup \tilde{F}$ is not f.g.o.s. in $\tilde{A}$.

Definition (2.2):

Let $\tilde{C}$ be a fuzzy set in a f.t.s. $(\tilde{A}, \tilde{T})$ and P_x^r is a fuzzy point of X. Then $\tilde{C}$ is called:

(a) Fuzzy generalized neighborhood (f.g.nbd) (resp. fuzzy neighborhood (f.nbd)of P_x^r if and only if there exists a fuzzy generalized open set (resp. fuzzy open set) $\tilde{B}$ in $(\tilde{A}, \tilde{T})$, such that $P_x^r \in \tilde{B} \subseteq \tilde{C}$.
(b) Fuzzy generalized quasi neighborhood (f.g.q.nbd)(resp. fuzzy quasi neighborhood (f.q.nbd) of P_x^r if and only if there exists a fuzzy generalized open set (resp. fuzzy open set) $\tilde{B}$ in $(\tilde{A}, \tilde{T})$, such that $P_x^r q \tilde{B} \subseteq \tilde{C}$.

Definition (2.3),(13):

A fuzzy subset $\tilde{f}$ of X × Y is said to be a fuzzy proper function from $\tilde{A} \in I^X$ to $\tilde{B} \in I^Y$ if

a. $\tilde{f}(x, y) \leq min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}, \forall (x, y) \in X \times Y$.
b. $\forall x \in X, \exists y_\circ \in Y$ such that $\tilde{f}(x, y_\circ) = \mu_{\tilde{A}}(x)$ and $\tilde{f}(x, y) = 0$ if $y \neq y_\circ$.

Definition (2.4),(13):

Let $\tilde{f}: \tilde{A} \rightarrow \tilde{B}$ be a fuzzy proper function, define the correspondent F-proper function $\tilde{f}: P(\tilde{A}) \rightarrow P(\tilde{B})$ and its reverse F-proper function $\tilde{f}^{-1}: P(\tilde{B}) \rightarrow P(\tilde{A})$ by

i. $\tilde{f}: P(\tilde{A}) \rightarrow P(\tilde{B}), \mu_{\tilde{f}(\tilde{C})}(y) = \sup\{\tilde{f}(x, y) \wedge \mu_{\tilde{C}}(x) : x \in X\}$, $\forall$ y $\in$ Yand $\forall \tilde{C} \in P(\tilde{A})$,
ii. $\tilde{f}^{-1}: P(\tilde{B}) \rightarrow P(\tilde{A}), \mu_{\tilde{f}^{-1}}(x) = \sup\{\tilde{f}(x, y) \wedge \mu_{\tilde{D}}(y) : y \in Y\}, \forall x \in X$ and $\forall \tilde{D} \in P(\tilde{B})$.

Proposition (2.1),(9):

Let $\tilde{f}: \tilde{A} \rightarrow \tilde{B}$ be a fuzzy proper function, if $\tilde{G} \in P(\tilde{B})$ is maximal, then $\tilde{f}^{-1}(\tilde{G}^c) = [\tilde{f}^{-1}(\tilde{G})]^c$.

Proposition (2.2),(9):

For fuzzy proper function $\tilde{f}: \tilde{A} \rightarrow \tilde{B}$. Then

i. $\tilde{C} \subseteq \tilde{f}^{-1}(\tilde{f}(\tilde{C})), \forall \tilde{C} \in P(\tilde{A})$.
ii. $\tilde{f}(\tilde{f}^{-1}(\tilde{D})) \subseteq \tilde{D}, \forall \tilde{D} \in P(\tilde{B})$.

3. Generalized Continuous Fuzzy Proper Functions:

In this section, a new type of fuzzy continuous functions which are called fuzzy generalized continuous functions are defined and their properties are studied.

Definition (3.1):

A fuzzy proper function $\tilde{f}$ from a f.t.s. $(\tilde{A}, \tilde{T})$ to f.t.s. $(\tilde{B}, \tilde{T})$ is said to be generalized continuous if $\tilde{f}^{-1}(\tilde{G})$ is f.g,o.s. in $\tilde{A}$, for each fuzzy open set $\tilde{G}$ in $\tilde{B}$.

Theorem (3.1):

If $\tilde{f}:(\tilde{A},\tilde{T}_1) \to (\tilde{B},\tilde{T}_2)$ is fuzzy continuous, then it is fuzzy generalized continuous.

Proof:

Let $\tilde{G}$ be a fuzzy open set in $\tilde{B}$, Since $\tilde{f}$ is fuzzy continuous function, then $\tilde{f}^{-1}(\tilde{G})$ is fuzzy open set in $\tilde{A}$. Let $\tilde{F} \subseteq \tilde{f}^{-1}(\tilde{G})$ such that $\tilde{F}$ is a fuzzy closed set in $\tilde{A}$, since $\tilde{f}^{-1}(\tilde{G})$ is fuzzy open set then $\tilde{f}^{-1}(\tilde{G})=int(\tilde{f}^{-1}(\tilde{G}))$ this implies $\tilde{F} \subseteq int(\tilde{f}^{-1}(\tilde{G}))$ hence $\tilde{f}^{-1}(\tilde{G})$ is fuzzy generalized open set in $\tilde{A}$, thus $\tilde{f}$ is fuzzy generalized continuous function.

Remark (3.1):

The converse of theorem (3.2) is not necessary true in general, as the following example

Example (3.1):

Let $X = \{a, b\}, Y = \{c, d\}, \tilde{T}_1 = \{\tilde{\emptyset}, \tilde{A}, \tilde{B}, \tilde{C}\}$ and $\tilde{T}_2 = \{\tilde{\emptyset}, \tilde{D}, \tilde{E}\}$,

Where $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ and $\tilde{E}$ are fuzzy sets defined by:

$\tilde{A} = \{(a, 0.9), (b, 0.6)\}$, $\tilde{B} = \{(a, 0), (b, 0.1)\}$,

$\tilde{C} = \{(a, 0.9), (b, 0.5)\}$, $\tilde{D} = \{(c, 0.9), (d, 0.8)\}$

$\tilde{E} = \{(c, 0.7), (d, 0.1)\}$. Define a fuzzy proper function $\tilde{f}:(\tilde{A},\tilde{T}_1) \to (D,\tilde{T}_2)$, by:

$\tilde{f} = \{((a,c), 0.9), ((a,d), 0), ((b,c), 0), ((b,d), 0.6)\}$

Then, $\tilde{f}$ is fuzzy generalized continuous function, but not fuzzy continuous function.

Theorem (3.2):

Let $\tilde{f}:(\tilde{A},\tilde{T}_1) \to (\tilde{B},\tilde{T}_2)$ be a fuzzy proper function, then if $\tilde{G}$ is maximal fuzzy closed subset of $\tilde{B}$ then $\tilde{f}$ is fuzzy generalized continuous function if and only if $\tilde{f}^{-1}(\tilde{G})$ is f.g.c.s., $\forall \tilde{G} \in \tilde{T}_2^c$.

Proof:

Let $\tilde{G}$ is maximal fuzzy closed set in $\tilde{B}$ with $P_y^r \in \tilde{G}$, then $\tilde{G}^c$ is
fuzzy open set Since $\tilde{f}$ is fuzzy generalized continuous function, then $\tilde{f}^{-1}(\tilde{G}^c)$ is f.g.o.s. in $\tilde{A}$.

Since $\tilde{G}$ is maximal fuzzy set , thus $\tilde{f}^{-1}(\tilde{G}^c) = [\tilde{f}^{-1}(\tilde{G})]^c$ this implies $[\tilde{f}^{-1}(\tilde{G})]^c$ is f.g.o.s. in $\tilde{A}$.

Hence $[[\tilde{f}^{-1}(\tilde{G})]^c]^c = \tilde{f}^{-1}(\tilde{G})$ is f.g.c.s. in $\tilde{A}$.

conversely Let $\tilde{G}$ be a maximal fuzzy closed set in $\tilde{B}$

Then $\tilde{f}^{-1}(\tilde{G})$ is f.g.c.s. in $\tilde{A}$ and $\tilde{G}^c$ is fuzzy open set in $\tilde{B}$

This implies that $[\tilde{f}^{-1}(\tilde{G})]^c$ is f.g.o.s. in $\tilde{A}$.

But $[\tilde{f}^{-1}(\tilde{G})]^c = \tilde{f}^{-1}(\tilde{G}^c)$, we have $\tilde{f}^{-1}(\tilde{G}^c)$ is f.g.o.s. in $\tilde{A}$

This implies $\tilde{f}$ is fuzzy generalized continuous function .

Theorem (3.3):

Let $\tilde{f}:(\tilde{A},\tilde{T}_1) \to (\tilde{B},\tilde{T}_2)$ be a fuzzy proper function, then the following statements are equivalent :

1. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each fuzzy open set $\tilde{G}$ in Y with $P_y^r \in \tilde{G}$, there exists f.g.o.s. $\tilde{H}$ in $\tilde{A}$, such that $P_x^r \in \tilde{H}$ and $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$.
2. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each fuzzy neighborhood $\tilde{N}$ of P_y^r in $\tilde{B}$, $\tilde{f}^{-1}(\tilde{N})$ is fuzzy generalized neighborhood of P_x^r in $\tilde{A}$.

Proof:

$(1 \Rightarrow 2)$ Let $\tilde{G}$ be a fuzzy open set in Y with $P_y^r \in \tilde{G}$

Let $\tilde{H}$ be f.g.o.s. in $\tilde{A}$, such that $P_x^r \in \tilde{H}$ and $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$.

Since $P_y^r \in \tilde{G}$ and $\tilde{G}$ fuzzy open set, thus $\tilde{G}$ is f.nbd of P_y^r.

Since $\tilde{H}$ is f.g.o.s. in I^X and $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$, then $\tilde{f}^{-1}(\tilde{f}(\tilde{H})) \subseteq \tilde{f}^{-1}(\tilde{G}) \Rightarrow$

$\tilde{H} \subseteq \tilde{f}^{-1}(\tilde{G})$, hence $P_x^r \in \tilde{H} \subseteq \tilde{f}^{-1}(\tilde{G})$.

Since $\tilde{H}$ is f.g.o.s. in $\tilde{A}$, then $\tilde{f}^{-1}(\tilde{G})$ is f.g.nbd of P_x^r in $\tilde{A}$.

$(2 \Rightarrow 1)$ suppose that $\tilde{G}$ is a fuzzy open set in $\tilde{B}$ with $P_y^r \in \tilde{G}$, then $\tilde{G}$ is f.nbd of $P_y^r \Rightarrow \tilde{f}^{-1}(\tilde{G})$ is f.g.nbd of P_x^r in I^X, this implies there exists f.g.o.s. $\tilde{H}$ in $\tilde{A}$ such that $P_x^r \in \tilde{H} \subseteq \tilde{f}^{-1}(\tilde{G})$.

Since $\tilde{H} \subseteq \tilde{f}^{-1}(\tilde{G}) \Rightarrow \tilde{f}(\tilde{H}) \subseteq \tilde{G}$.

Theorem (3.4):

Let $\tilde{f}:(\tilde{A},\tilde{T}_1) \to (\tilde{B},\tilde{T}_2)$ be a fuzzy proper function, then the following statements are equivalent:

1. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each f.nbd $\tilde{M}$ of P_y^r in $\tilde{B}$, $\tilde{f}^{-1}(\tilde{M})$ is f.g.nbd of P_x^r in $\tilde{A}$.
2. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each f.nbd $\tilde{G}$ of P_y^r in $\tilde{B}$, there exists f.g.nbd $\tilde{H}$ of P_x^r in $\tilde{A}$, such that $P_x^r \in \tilde{H}$ and $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$.

Proof:

$(1 \Rightarrow 2)$ Let $\tilde{G}$ f.nbd of P_y^r in $\tilde{B}$, then $\tilde{f}^{-1}(\tilde{G})$ is f.g.nbd of P_x^r in $\tilde{A}$ this implies that
$\tilde{f}(\tilde{f}^{-1}(\tilde{G})) \subseteq \tilde{G}$.

Let $\tilde{H} = \tilde{f}^{-1}(\tilde{G})$ then there exists f.g.nbd of P_x^r in $\tilde{A}$ such that $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$.

$(2 \Rightarrow 1)$ suppose that $\tilde{M}$ is f.nbd of P_y^r in $\tilde{B}$, then there exists
f.g.nbd $\tilde{H}$ of P_x^r in $\tilde{A}$ such that $\tilde{f}(\tilde{H}) \subseteq \tilde{M}$ this implies that $\tilde{H} \subseteq \tilde{f}^{-1}(\tilde{M})$.

Since $\tilde{H}$ is f.g.nbd of P_x^r in $\tilde{A}$, then $\exists$ f.g.o.s. $\tilde{G}$ in $\tilde{A}$ such that
$P_x^r \in \tilde{G} \subseteq \tilde{H} \subseteq \tilde{f}^{-1}(\tilde{M})$, hence $P_x^r \in \tilde{G} \subseteq \tilde{f}^{-1}(\tilde{M})$ thus $\tilde{f}^{-1}(\tilde{M})$ is
f.g.nbd of P_x^r in $\tilde{A}$.

Theorem (3.5):

Let $\tilde{f}:(\tilde{A},\tilde{T}_1)\to(\tilde{B},\tilde{T}_2)$ be a fuzzy proper function, then the following statements are equivalent :

1. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each f.nbd $\tilde{G}$ of P_y^r in $\tilde{B}$, there exists f.g.nbd $\tilde{H}$ of P_x^r in $\tilde{A}$, such that $P_x^r \in \tilde{H}$ and $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$.
2. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each fuzzy open set $\widetilde{M}$ in $\tilde{B}$ such that $P_y^r q \widetilde{M}$, there exists f.g.o.s. $\tilde{N}$ in $\tilde{A}$, such that $P_x^r q\, \tilde{N}$ and $\tilde{f}(\tilde{N}) \subseteq \widetilde{M}$.

Proof:

$(1 \Rightarrow 2)$ Let P_x^r be a fuzzy point in $\tilde{A}$ and $\widetilde{M}$ be a fuzzy open set in $\tilde{B}$ such that $P_y^r\, q\, \widetilde{M}$, then $\mu_{\widetilde{M}}(y) > \mu_{(P_y^r)^c}(y)$ this implies $(P_y^r)^c \subseteq \widetilde{M}$.

Since any fuzzy set is the union of all its fuzzy points, then $(P_y^r)^c = \cup_{i\in I} P_{i_y}^{r_i}$

Such that $r_i = \mu_{\tilde{B}}(y_i) - r$, thus $\cup_{i\in I} P_{i\,y}^{r_i} \subseteq \widetilde{M} \subseteq \widetilde{M}$, this implies that $P_{i_y}^{r_i} \in \widetilde{M} \subseteq \widetilde{M}$, $\forall i$

Since $\widetilde{M}$ be a fuzzy open set, then $\widetilde{M}$ be a f.nbd of $P_{i_y}^{r_i}$ this implies that there exists f.g.nbd. $\widetilde{H}$ of $P_{i_x}^{r_i}$ in $\tilde{A}$, such that $P_{i_x}^{r_i} \in \widetilde{H}$, $\tilde{f}(\tilde{H}) \subseteq \widetilde{M}$ and $r_i = \mu_{\tilde{A}}(x_i) - r$.

Since $\widetilde{H}$ is f.g.nbd. of $P_{i_x}^{r_i}$, then there exists f.g.o.s. $\tilde{N}$ in $\tilde{A}$ such that $P_{i_x}^{r_i} \in \tilde{N} \subseteq \tilde{H}$, from $\tilde{N} \subseteq \tilde{H}$ we have $\tilde{f}(\tilde{N}) \subseteq \tilde{f}(\tilde{H}) \subseteq \widetilde{M}$, thus $\tilde{f}(\tilde{N}) \subseteq \widetilde{M}$, and from $P_{i_x}^{r_i} \in \tilde{N}$, $\forall i$ we have $\cup_{i\in I} P_{i_x}^{r_i} \subseteq \tilde{N}$.

Since $\cup_{i\in I} P_{i_x}^{r_i} = (P_x^r)^c$ this implies that $(P_x^r)^c \subseteq \tilde{N}$, therefore $\mu_{\hat{A}}(x_i) - r \le \mu_{\tilde{N}}(x)$

Thus $P_x^r q\, \tilde{N}$.

$(2 \Rightarrow 1)$ Let $\widetilde{M}$ be a fuzzy open set in $\tilde{B}$ such that $P_y^r q\, \widetilde{M}$, then

$(P_y^r)^c = \cup_{i\in I} P_{i_y}^{r_i} \subseteq \widetilde{M} \subseteq \widetilde{M}$ this implies $\widetilde{M}$ is f.nbd of $P_{i_y}^{r_i}$.

Since $\widetilde{M}$ be a fuzzy open set in $\tilde{B}$ such that $P_y^r q\, \widetilde{M}$, then there exists f.g.o.s.

$\tilde{N}$ of P_x^r such that $P_x^r q\, \tilde{N}$ and $\tilde{f}(\tilde{N}) \subseteq \widetilde{M}$, from $P_x^r q\, \tilde{N}$ we have

$(P_x^r)^c = \cup_{i\in I} P_{i_x}^{r_i} \subseteq \tilde{N} \subseteq \tilde{N}$, hence $\tilde{N}$ is f.g.nbd of $P_{i_x}^{r_i}$ and $\tilde{f}(\tilde{N}) \subseteq \widetilde{M}$.

Theorem (3.6):

Let $\tilde{f}:(\tilde{A},\tilde{T}_1)\to(\tilde{B},\tilde{T}_2)$ be a fuzzy proper function, then the following statements are equivalent :

1. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each fuzzy open set $\tilde{G}$ in $\tilde{B}$ such that $P_y^r q \tilde{G}$, there exists f.g.o.s. $\tilde{H}$ in $\tilde{A}$, such that $P_x^r q\, \tilde{H}$ and $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$.
2. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each f.q.nbd. $\widetilde{M}$ of P_y^r, $\tilde{f}^{-1}(\widetilde{M})$ is f.g.q.nbd. of P_x^r.

Proof:

$(1 \Rightarrow 2)$ Let P_x^r be a fuzzy point in $\tilde{A}$ and $\widetilde{M}$ be a f.q.nbd. of P_y^r in $\tilde{B}$.

Then there exists a fuzzy open set $\tilde{N}$ in I^Y such that $P_y^r q\, \tilde{N} \subseteq \widetilde{M}$.

Since $\tilde{N}$ is fuzzy open set in $\tilde{B}$ and $P_y^r q\, \tilde{N}$, then there exists

f.g.o.s. $\tilde{H}$ in $\tilde{A}$ Such that $P_x^r q\, \tilde{H}$ and $\tilde{f}(\tilde{H}) \subseteq \tilde{N} \subseteq \widetilde{M}$ that is $\tilde{f}(\tilde{H}) \subseteq \widetilde{M}$ thus $\tilde{H} \subseteq \tilde{f}^{-1}(\widetilde{M})$

this implies that $P_x^r q\, \tilde{H} \subseteq \tilde{f}^{-1}(\widetilde{M})$, therefore $\tilde{f}^{-1}(\widetilde{M})$ is f.g.q.nbd. of P_x^r.

$(2 \Rightarrow 1)$ Suppose that $\tilde{G}$ be a fuzzy open set in $\tilde{B}$ such that $P_y^r q\, \tilde{G}$

Since $P_y^r q\, \tilde{G} \subseteq \tilde{G}$, then $\tilde{G}$ is f.q.nbd of P_y^r this implies that $\tilde{f}^{-1}(\tilde{G})$ is f.g.q.nbd of P_x^r so

$\exists$ f.g.o.s. $\tilde{H}$ in $\tilde{A}$ such that $P_x^r q\, \tilde{H} \subseteq \tilde{f}^{-1}(\tilde{G})$ thus $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$.

Theorem (3.7):

Let $\tilde{f}:(\tilde{A},\tilde{T}_1)\to(\tilde{B},\tilde{T}_2)$ be a fuzzy proper function, then the following statements are equivalent :

1. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each f.q.nbd. $\tilde{G}$ of P_y^r, $\tilde{f}^{-1}(\tilde{G})$ is f.g.q.nbd. of P_x^r.
2. For each fuzzy point P_x^r in $\tilde{A}$, for each $y \in Y$ and for each f.q.nbd. $\widetilde{M}$ of P_y^r, there exists f.g.q.nbd. $\tilde{N}$ of P_x^r such that $\tilde{f}(\tilde{N}) \subseteq \widetilde{M}$.

Proof:

$(1 \Rightarrow 2)$ Let P_x^r be a fuzzy point in $\tilde{A}$ and $\widetilde{M}$ be a f.q.nbd . of P_y^r in $\tilde{B}$.

Then $\tilde{f}^{-1}(\widetilde{M})$ is a f.g.q.nbd. of P_x^r, let $\tilde{N} = \tilde{f}^{-1}(\widetilde{M})$ so $\tilde{f}(\tilde{N}) = \tilde{f}(\tilde{f}^{-1}(\widetilde{M})) \subseteq \widetilde{M}$.

$(2 \Rightarrow 1)$ Suppose that $\tilde{G}$ is f.q.nbd of P_y^r

Then there exists f.g.q.nbd $\tilde{H}$ of P_x^r such that $\tilde{f}(\tilde{H}) \subseteq \tilde{G}$ so $\tilde{H} \subseteq \tilde{f}^{-1}(\tilde{G})$

Since $\tilde{H}$ is f.g.q.nbd of P_x^r, then there exists f.g.o.s. $\tilde{E}$ in $\tilde{A}$ such that

$P_x^r q\, \tilde{E} \subseteq \tilde{f}^{-1}(\tilde{G})$ thus $\tilde{f}^{-1}(\tilde{G})$ is f.g.q.nbd of P_x^r.

Theorem (3.8):

If $\tilde{f}:(\tilde{A},\tilde{T}_1)\to(\tilde{B},\tilde{T}_2)$ is fuzzy generalized continuous function and $\tilde{g}:(\tilde{B},\tilde{T}_2)\to(\tilde{C},\tilde{T}_3)$ is fuzzy continuous function, then $\tilde{g}o\tilde{f}:(\tilde{A},\tilde{T}_1)\to(\tilde{C},\tilde{T}_3)$ is fuzzy generalized continuous function.

Proof:

Let $\tilde{G}$ be a fuzzy open set in Z.

Since $\tilde{g}$ is fuzzy continuous function, then $\tilde{g}^{-1}(\tilde{G})$ is fuzzy open set in Y

Since $\tilde{f}$ is fuzzy generalized continuous function, then $\tilde{f}^{-1}(\tilde{g}^{-1}(\tilde{G}))$ is f.g.o.s. in X, from $(\tilde{g}o\tilde{f})^{-1}(\tilde{G}) = \tilde{f}^{-1}(\tilde{g}^{-1}(\tilde{G}))$ we have $\tilde{g}o\tilde{f}: X \to Z$ is fuzzy generalized continuous function.

Remark (3.2):

Theorem (3.8) is not necessary true in general if $\tilde{g}$ is fuzzy generalized continuous

Example (3.2):

Let $X = \{a, b\}, Y = \{c, d\}$ and $Z = \{e, f\}$. fuzzy sets $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F}$ and $\tilde{G}$ are defined as:

$\tilde{A}$ = {(a, 0.6), (b, 0.6)}, $\tilde{B}$ = {(a, 0), (b, 0.1)}

$\tilde{C}$ = {(c,0.7), (d, 0.6)}, $\tilde{D}$ = {(c, 0.2), (d, 0.3)}

$\tilde{E}$ = {(e, 0.7), (f, 0.7)}, $\tilde{F}$ = {(e, 0.4), (f, 0.4)}

$\tilde{G}$ = {(e, 0.1), (f, 0.7)}.

Let $\tilde{T}_1 = \{\tilde{\emptyset}, \tilde{A}, \tilde{B}\}$, $\tilde{T}_2 = \{\tilde{\emptyset}, \tilde{C}, \tilde{D}\}$ and $\tilde{T}_3 = \{\tilde{\emptyset}, \tilde{E}, \tilde{F}\}$.

Then the fuzzy proper functions $\tilde{f}: (\tilde{A}, \tilde{T}_1) \to (\tilde{D}, \tilde{T}_2)$ and $\tilde{g}: (\tilde{D}, \tilde{T}_2) \to (\tilde{F}, \tilde{T}_3)$ where $\tilde{f} = \{((a, c), 0.6), ((a, d), 0), ((b, c), 0), ((b, d), 0.6)\}$ and $\tilde{g} = \{((c, e), 0),, ((c, f), 0.7), ((d, e), 0.6), ((d, f), 0)\}$ are fuzzy generalized continuous functions but $\tilde{g}o\tilde{f}$ is not fuzzy generalized continuous function.

Conflicts of Interest: None.

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تعميم حول الدالة الفعلية الضبابية المستمرة من فضاء تبولوجي ضبابي الى فضاء تبولوجي ضبابي آخر

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الخلاصة:

الهدف من هذا البحث هو تقديم ودراسة مفاهيم المجموعة المفتوحة العامة الضبابية، المجموعة المغلقة العامة الضبابية، الدالة المستمرة العامة الضبابية وأثبات النتائج حول هذه المفاهيم.

الكلمات المفتاحية: المجموعات المغلقة العامة الضبابية، الدالة الفعلية الضبابية، الدوال الفعلية المستمرة العامة الضبابية.