On Singular Sets and Maximal topologies

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Abstract:
In this Work , we study The concept of maximal topologies and its relation with Singular sets , furthermore we study the spaces which are maximal with respect to semi- regular property and we proved that if τ is sub maximal has property P then τ is maximal P if and only if τ is non singular (with respect to P) we prove that if P is contractive, semi – regular and τ is non Singular (with respect to P) then every τ_s- Singular set V $\cup$\{x\} such that x $\in Cl_{\tau^*}V_i - Int_{\tau^*}V_i$ is τ_s-open and we provide some theorems.

الملخص:
درسنا في هذا البحث التوبولوجيات الاعظمية وعلاقتها بالمجموعات المنفردة بالإضافة إلى ذلك درسنا الفضاءات الاعظمية المعتمدة على خاصية شبه منتظم وبرهنا أذا كان τ اعظمي جزئي يمتلك الخاصية P فأن τ اعظمي P اذا وفقط اذا τ ليس منفرداً (بالاعتماد على الخاصية P) وبرهنا أذا كان P شبه منتظم، τليس منفرداً (بالاعتماد على الخاصية P) فان كل مجموعة منفردة - τ_s و V$\cup$\{x\} ، بحيث $x\in Cl_{\tau^*}V_i - Int_{\tau^*}V_i$ تكون مفتوح - τ_s وبرهنا بعض المبرهنات الأخرى.

1- Introduction:

The family of all topologies definable on an infinite set X is ordered by inclusion which is denoted by LT (X). A member τ of LT (X) is said to be Maximal with respect to p if τ has property p but no stronger member of LT (X) has property p. Recall that a τ -open set V is τ -regular open if V=int_τ cl_τ V. The topology generated by the family τ -regular open sets is called semi- regularization of τ and denoted by τ_s. A topological property p is called semi-regular when τ ∈LT(X) is P if and only if τ_s ∈LT (X) is P. Hausdorff and connectedness are the classic examples of semi-regular properties given τ∈LT (X) and a subset V of X the boundary of $V, cl_\tau V - int_\tau V$ is denoted by Ψ_τV, if D is a family of subsets of X, the topology generated T∪D is denoted by <T∪D>, when D={A} for some A⊆X we write <T∪D> as T(A).

The concept of maximal topologies was first introduced in 1943 by E. Hewitt when he showed that compact Hausdorff spaces are maximal compact In 1948 A.Ramanathan proved that a topological subsets are precisely the closed sets, In 1977 Guthrie and Stone introduced the concept of singular set to construct a maximal connected expansion of the real line. In 1986 Neumann-Lara and Wilson generalized the notion of a singular set to characterize T1 maximal connected spaces.

2 Preliminaries

Definition2.1[4]

Let (X, τ) b a o olo ical s ac ⊆ X, A and the intersection of all closed super sets of A is called the closure of A which is denoted by Cl (A).

Definition2.2[4]

Let (X, τ) b a o olo ical s ac ⊆ X, A and A point x∈X is said to be an interior point of A if and only if A is a neighborhood of x.

The set all interior points of A is called the interior of A which is denoted by Int(A).

Definition2.3[4]

Let τ_1 and τ_2 b wo o olo i s fo a s X, τ_1 is weaker a τ an τ_2 is st on τ_1 if and only if v τ_1-o n s τ_2- is τ open set.

Definition2.4[4]

Let (X, τ) b a o olo ical s ac and A⊆X, we say that A is regular open set if and only if A= Int (Cl(A)).

Definition2.5[4]

A o olo ical s ac (X τ) is said to be semi- regular space if and only if every open set is union for regular open sets.

Definition2.6[4]

A o olo ical s ac (X τ) is said o b la if and only if fo every closed F and every P∉F there are disjoint open sets G and H in X such that F⊂G, P∈H.

Definition2.7[4]

A o olo ical s ac (X τ) is said o b disconn c d if and only if there are disjoint open sets G and H in X such that X= G∪ H, when no such disconnection exists, X is connected.

Definition2.8[4]

Let (X, τ) b a o olo ical space and A⊆ X, we say that A is a singular set if either A is regular open or there exists x∈A such that A-{x} is regular open.

3 Singular sets and maximal topologies

Definition 3.1:[3]

Given τ ∈LT(X), τ is sub maximal if every τ -dense set is τ -open.

Theorem 3.1[4]:

Given $T \in LT(x)$, the following statements are equivalent.

1) τ is sub maximal
2) The family of τ - dense open sets is an ultra filter of τs- dense sets.
3) For any $\alpha \in LT(X)$ such that $\tau \subset \alpha$, $\alpha_s \neq \tau_s$.
4) Every subset of X is the union of an open set and a closed set.
5) For every subset A of X which is not open, there are non empty proper closed sets B_1, B_2 such that $B_1 \subseteq A \subseteq B_2$
6) Every subset of X is the intersection of an open and a closed set.
7) Every subset A of X, for which int $A=\phi$ is closed.
8) Every subset A of X, for which int $A=\phi$ is discrete
9) cl (A)-A is closed, for every subset A of X
10) cl (A)- A is discrete, for every subset A of X

Proof:[4]

Lemma 3.1:

If $\tau \in LT(X)$ is sub maximal and $B \subseteq X$ then $(int_\tau\, cl_\tau\, B) \cup \{x\}$ is $\tau(B)$-open, for all $x \in B\text{-}int_\tau B$.

<u>Proof:</u>

since $(X\text{-}B) \cup (int_\tau B) \cup \{x\}$ is τ -dense, so by hypothesis is τ -Open. Now

$(int_\tau B) \cup \{x\} = B \cap [(X\text{-}B) \cup (int_\tau B) \cup \{x\}]$ and so is $\tau(B)$ -open thus $(int_\tau cl_\tau int_\tau B) \cup \{x\}$ is $\tau(B)$-open

Definition 3.2[4]

Give $\tau \in LT(X)$ has property P, V is t- regular open and $x \in X$, then $V \cup \{x\}$ is said to be a τ- singular (with respect to P) set at x, if $\tau(V \cup \{x\})$ has property P.

Example 3.1:

consider the real line with usual topology let V be the following union of open intervals $(-1,0) \cup \left\{\bigcup_{n=1}^{\infty}\left(\frac{1}{2n+1},\frac{1}{2n}\right)\right\}$ then V∪{0} is a singular (with respect to connectedness) set at 0, but is not an open set.

Definition 3.3[4]

Give $\tau \in LT(X)$, t is called non-singular (with respect to p) if τ has property P and every singular (with respect to P)set is τ - open.

Theorem 3.2:

let $\tau \in LT(X)$ is sub maximal and P, if τ is maximal P then τ is non singular (with respect to P).

Proof:

suppose τ is P but not maximal P. then there is a set $B \subset X$ such that $\tau \subset \tau(B)$. so there is a point $x \in B - int_\tau B$. Now $V = int_\tau cl_\tau B$ is τ-regular open, since τ is sub maximal and $int_\tau B \cup \{x\} = (V \cup \{x\} \cap [int_\tau B \cup (X - cl_\tau B) \cup \{x\}]$ then V is not τ-open. But by lemma 1, V∪{x} is τ(B)-open and so any weaker than τ(B) has property P, τ(V∪{x}) is P that is V∪{x} is a τ-singular (with respect to P) set which is not τ-open.

Lemma 2.3:

Suppose $\tau \in LT(X)$ is P, $A \subseteq X$ and β_x is a filter base of τ-singular (with respect to p) sets at x, when $x \in X$. let $\tau^* = \langle \tau \cup \beta_x \rangle$ then the τ^*- closure of A is described by

$$\bar{A}^* = \begin{cases} \bar{A} \ if \ x \in \overline{(B-\{x\}) \cap A} \ for \ every \ B \in \beta_x \\ \bar{A} - \{x\} if \ x \notin \overline{(B-\{x\}) \cap A} \ for \ every \ B \in \beta_x \end{cases}$$

Proof:

Let $y \in \bar{A} - \{x\}$ then a $(\tau^* - \tau)$ neighborhood of y contains a set of the form $G \cap B$ when $y \in G \in \tau$ and $y \in B \in \beta_x$. by definition of a singular set, either B or B-{x} is τ-regular open so that G∩B is τ-neighborhood of y but y∈ $\bar{A}$, so G∩B∩A=ϕ that is y∈ $\bar{A}^*$ Hence $\bar{A} - \{x\} \subseteq \bar{A}^* \subseteq \bar{A}$. finally it is clear that x∈$\bar{A}^*$ if and only if $X \in \left(\overline{B - \{x\} \cap A}\right)$ for every B∈β_x.

Lemma 3.3[4]

suppose τ∈LT(X) is P and β_x is a filter base of τ-singular (with respect to P) sets at x, where x∈X. let τ^*=<τ∪β_x> if G∈ τ^* and x∉G then G∈τ.

Definition 3.4[4]

A topological property P is called contractive if for a given member τ of LT (X) with property P any weaker member of LT (X) has property P.

Lemma 3.4:

suppose τ∈LT(X) has property P and that every singleton τ-Singular Set is τ-open, while β_x is an ultra filter of τ-singular (with respect toP) sets at x, where x∈X, let τ` =< τ ∪$\beta_x$> if τ` has property P then every τ`- singular set at x is τ`- open.

Proof:

Suppose Y ∪{x} is τ`- singular at x but is not τ`- open, so we assume that V is τ`- regular open and the x∈ $\Psi_{\tau'}$V, by lemma 3,V is τ-open and by lemma 3.2 $cl_{\tau'}$V= $cl_\tau$V, since τ ⊆ τ`, V⊆ $int_{\tau'1}$ V ⊆ $int_{\tau'}$ $cl_{\tau'}$ V=V and therefore V is τ-regular open. Now for each B∈β_x, B∪ (V∪{x}) ≠ϕ because x∈ $\Psi_{\tau'}$V and also τ(B∩(V∪{x}) ⊆ τ' (V∪{x}) But p is contractive and the intersection of any two regular open sets is regular open thus V∪ {x} meets eac m mb x of β τ-singular set at x, since $V\cup\{x\}\notin \tau'$ then $V\cup\{x\}\notin \beta_x$ that is β_x is not an ultra filter of τ-singular sets at x.

Theorem 3.3:

Suppose P is a semi – regular property and that $\tau\in LT(X)$ is p and every singleton τ-singular set is τ-open. Let D be an ultra filter of τ-dense sets. Given $x\in X$, l $_x$ be anltra filter of τ-singular (with respect to P) sets of x. Let $\tau`= <\tau\cup D\cup (\cup_{x\in X} \beta_x)>$ is $\tau`$ has property p, then $\tau`$ is a maximal P.

Proof:

Let $\tau^*=<\tau\cup D>$ which is sub maximal so $\tau`$ is sub maximal suppose $V\cup\{x\}$ is $\tau`$ – singular at x but is not $\tau`$-open. As every singleton t-singular set τ-open is $B\cup\{x\}\in \beta_x$ then $x\in cl_\tau B$ and so $x\in cl_\tau B$ thus $int_{\tau^*}V$ is τ^*- regular open and so must be τ-regular open, Now τ^* is sub maximal and $(int_{\tau^*}V)\cup\{x\}=(V\cup\{x\})\cap[((int_{t^*}V)\cup\{X-V\}\cup\{x\}]$ we have $<\tau^*\cup\beta_x\cup \{(int_{t^*}V)\cup\{x\}\}> \subseteq \tau` (V\cup\{X\})$ But P is contractive, and $V\cup\{x\}$ is $\tau`$-singular, so $<\tau\cup \beta_x\cup\{(int_{\tau^*}V)\cup\{x\}>$ is P, Now $(int_{t^*}V)\cup\{x\}$ can not be $<\tau\cup \beta_x>$- regular open (other wise, $V\cup\{X\}$ is τ^*-open) so by lemma 3.3 int τ^*V is $<\tau\cup \beta_x>$- regular open and there fore $(int\ \tau^*\ V)\cup\{x\}$ is $<\tau\cup \beta_x>$- singular set at x, which is not $<\tau\cup \beta_x>$-open (since $V\cup\{x\}$ is not $\tau`$-open) which is a contradiction with lemma 3.4

Theorem 3.4 :

Suppose P is contractive, semi-regular, and that $\tau\in LT(X)$ is non singular (with respect to P), then every τ_S singular set $V\cup\{x\}$ such that $x \in \Psi_{\tau*} V$ is τ_s - open.

Proof:

Suppose τ_s (V∪{X}) has property P where V is τ_s −regular open and x ϵ $\Psi_{\tau s}$ V, V is τ- regular open and $\Psi_{\tau s}$V= Ψ_τV now τ =< τ_s ∪D>, where D is a filter base of τ_s(V∪{x})- dense sets , and because P is semi- regular

<τ_s ∪D∪{V∪ {x} > = τ(V∪{x}) is also P, Hence V∪ {x}is τ- singular at x , and so by hypothesis is τ- open but x∈ ΨV so x ∈V, that is V ∪{x} =V∈ τ_s

Definition 3.5 [6]

τ is feebly compact (Quasi – H – closed) if every countable open filter base has a cluster point.

Definition 3.6 [6]

let h∈X we say that h is an almost H- point if there is accountable filter base of non empty τ - regular open sets such that {h} = ∩ { Cl_τW: W∈ Ŵ}

Definition 3.7 [6]

A topological Space (X, τ) is an almost H – space (almost E_1 – space) if every point is an almost H – point (almost E_1- point).

Theorem 3.5 :

Suppose τ∈LT(X) is feebly compact if V is a τ - regular open and x is non – isolated in the subspace X-V Then V ∪ {x} is not singular if and only if x is an almost H-point (almost E_1-point) in the Subspace X-V.

<u>proof</u>:

Let $\tau^* = \tau(V \cup \{X\})$ is not feebly compact if and only if There is a τ^*- open filter base $\zeta = \{ G_i : i \in I\}$ such that $\cap \{cl\tau^* G_i : i \in I \} = \phi$. Now that is some $G \in \zeta$ such that $x \notin G$, and so for any $i,j \in I$, $G_i \cap G_j \neq \{x\}$ (other wise $G \cap G_i \cap G_j = \phi$) By lemma 3.3 for each $i \in I$, $G_i - \{x\} \subseteq int_\tau G : \subseteq Gi$, so $\zeta = \{ int\, \tau G_i : i \in I\}$ in a filter base of τ- open sets , But τ is feebly compact so ζg, Then there is as et $G_0 \in \zeta$ Such that $h \in (cl\tau\, G_0) - (cl\tau^* G_0$, so by Lemma 3.4 h=x and there is a τ - nieghbour hood N of x Such that $N \cap V \cap G_0 = \phi$ Now G_0 Since $x \in cl\tau\, G_0$,and because V is τ - regular open ,$G_0 \cap (X - cl\tau\, V) \neq \phi$,if follows that for all $i \in I$,$(int_\tau\, G_i) \cap (X - cl_\tau\, V) \neq \phi$ and so that $H = \{(int_\tau G_i) \cap (X - cl\tau\, V) \neq \phi : i \in I\}$ is a τ - open filter base and that x is the only τ- cluster point of H furthermore $\{x\} = \cap \{cl\tau\, int\tau\, cl\tau\, [G_i \cap (X - cl\tau\, V)] : i \in I\}$ so that x is an H-point .

The main result

1) If P is a contractive semi-regular property then a maximal P topology is sub maximal.
2) Given $\tau \in LT(X)$ is sub maximal and has property P then τ is non singular if τ is maximal P.
3) If $\tau` = <\tau \cup D \cup (\cup_{x \in X} \beta_x)>$ where $\tau$ a o olo y as o y P, D b an l a fil - dense sets of $\tau$ and $\beta_x$ b an l a fil – of $\tau$ sin la (wi s c` is a maximal P expansion of τ.

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