Traffic Flow Problem with Differential Equation (Mathematical Model)

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Abstract:

We modify the non-linear car following model. The modification is generally aimed at improving the validity of the model.

Keywords: traffic flow Problem; conservation of cars; a velocity-density relationship; concentration of vehicles

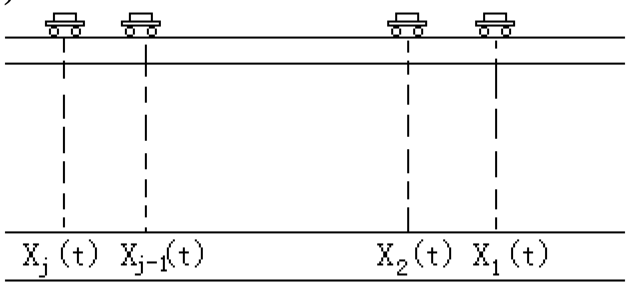
AMS Subject Classifications (2000): Primary 90B20, 90B06, 60K30; Second 35L65

CLC. O175.1 **Document: A**

1. Introduction

Car following theory is a model for the motion of a number of individual vehicles as they proceed down along straight road with no passing allowed. Many mechanical physiological aspects of the deriver-vehicle-road systems make modeling difficult. However we can assume that we have a line of n perfect derivers, in the sense that each driver can follow eaxactly the low of motion which pertinent.

Now assume the position of vehicle **j** at time **t** is given by $x_j(t)$, see figure(1) below



Fig(1) Position of the Vehicles

A good rule of thumb for following another vehicle at a safe distance is allowing you the length of a car for ten meter per hour (14.67) m/s.

When we are travelling in terms of function $x_j(t)$, we have

$$x_{j-1}(t) - x_j(t) = l\,\frac{dx_j/dt}{14.67} + l$$

1.A

Where l =length of the car (normaly of 15 feet) and $\frac{dx_j}{dt}$ =speed of **j**th vehicle.

Length of a vehicle, must be added to the separation between vehicles to give the spacing

$$x_{j-1}(t) - x_j(t)$$

We assume that each driver is able to follow (1.A) at each time **t**.

This means that the driver must accelerate (or decelerate) his vehicle according to

$$\frac{d^2x_j}{dt^2} = \lambda\left(\frac{dx_{j-1}}{dt} - \frac{dx_j}{dt}\right)$$

1.B

Where $\lambda = \frac{14.67}{l} \approx 1$ per sec.

A driver must observe the relatve velocity between his vehicle and the vehicle ahead, and adjust his acceleration according to that case.

2. (a) Traffic flow theory

Traffic flow theory involves the development of mathematical relationships among the primary elements of a traffic stream: Flow density and speed.

These aspects of traffic flow theory can be used in the planning, design, and operation of highway system^{s}[1].

In the present paper our plan is to use differential equation to deal with the "traffic flow problems".

The traffic flow has been discussed at length, which includes linear & non-linear car following with or without delay. The resulting models are used to predict the jam concentration and free and maximum traffic flow.

(b) Traffic flow models

We consider an arbitrary section of roadway between $x=a$ and $x=b$, if there are neither entrances nor exits on this segment of the road, then the number of cars between $x=a$ and $x=b$ ($N=\int_a^b P(x,t)dx$) the integral from of conservation of cars is

$$\frac{d}{dt}\int_a^b P(x,t)dx = q(a,t) - q(b,t) .$$

Thus, we making it a partial derivative, it follows that $\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$, we call it conversation of cars[2].

A car-following model, Typical of the equation considered is $\frac{dv}{dt} = k\frac{\Delta v}{\Delta x}$, where v is speed, Δv is speed difference, andΔx is spacing[3], when all vehicles are equally spaced and at same speed that is acceleration. so that from any known pair(v_0, Δx_0), one may writes $v-v_0=k\ln\frac{\Delta x}{\Delta x_0} = k\ln\frac{D_0}{D}$, where D is density, $D=1/\Delta x$. Denoting flow as Q, noting $Q=VD$, and observing V=0 at jam density D_j, that $Q=KD\ln\frac{D_j}{D}$ [3].

In addition to the car-following derivations of macroscopic flow relationships, such expressions have been arrived at by curve-fitting hypotheses[4] by observation of safe headways[5], by heat-flow analogies[6], and by fluid-flow analogies[6], the heat and fluid analogies center on equilibrium conditions for partial differential equation expressing a heat or mass balance (equation of continuity) the derivation on safe headways allows for a space headway which includes a vehicle length L, a reaction-time distance C_1V (where C_1 is reaction of "dead" time) and a deceleration distance C_2V^2 (where C_2 is determined by braking capability)[7]

$$\Delta x = L + C_1 V + C_2 V^2 \text{ or since Q=VD=V/X}$$

$$Q = \frac{V}{L + C_1 V + C_2 V^2}$$

In addition to the macroscopic Kerner-Korhausor model, this model includes the continuity equation[8,9]

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0$$

(c) Nonlinear car-following laws

Several modifications of the linear car-following model known collectively as nonlinear car-following models modify the assumption that the response sensitivity λ is a constant and the same for all drivers. A useful definition in this discussion is the car spacing $P_n(t)$ between two successive cars $P_n(t) \equiv X_n(t) - X_{n+1}(t)$. step function law. Assuming the step function for λ:

$$\lambda = \lambda(P_n) = \begin{cases} \alpha & if \quad 0 < P_n \le P^* \\ \beta & if \quad P_n > P^* \end{cases},$$ α and β are constants[10].

But, in the present CFT (car-following theory) used different sensitivities for accelaration and deceleration, in addition to reciprocal spacing law: Edi's law[11]

$$\frac{dx_{n+1}^2(t+T)}{dt^2} = \lambda_0 \frac{\left(\frac{dx_{n+1}(t+T)}{dt}\right)\left(\frac{dx_n(t)}{dt} - \frac{dx_{n+1}(t)}{dt}\right)}{(x_n(t) - x_{n+1}(t))^2}$$

(d)A Velocity-Density Relationship.

The two varibles, traffic density and car velocity, are related by only one equation, conservation of car, $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$

The density depends on the velocity. If we assume that under all circumstances the driver's velocity is a known function of ρ, determined by $u = u(p)$, then the conservation of cars implies[12]

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u(\rho)) = 0$$

3.Basic Equations——

The basic conservation for traffic flow can be derived as follows[13-16].

Let us consider the flows of vechicles on along road are compared with the averages distances between vhicles.

Let N be the number of vehicles between point X and X+ΔX on the road at time. We shall assume that $\exists \rho(x,t)$ s.t for any X, ΔX and t

$$\rho(x,t) = \lim_{\Delta X \to 0} \frac{N}{\Delta X}$$

3.A

If ρ is continuous. ρ is the number of vehicles per unit length in the infinitesmal length between X and X+ΔX at time t.

Empirical values of ρ can be determined from aerial photograph of the road

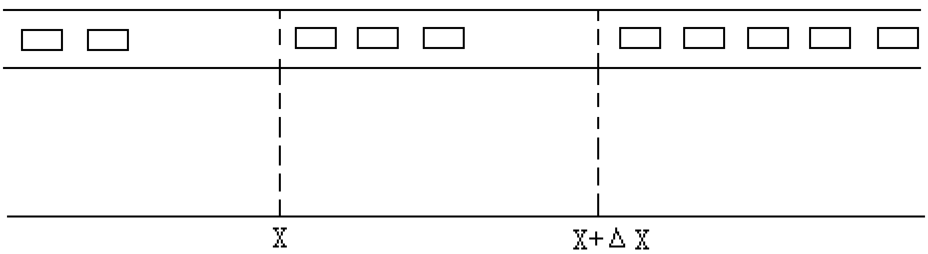


Fig.2 the traffic situation at some time t

Now we define the flow rate q is $q(x,t) = \lim_{\Delta t \to 0} \frac{Q}{\Delta t}$ where Q is the total number of vhicles crossing point X in the time lap Δt

Let us consider the conservation low of vehicles in the road.

Thus the conservation of vechicles requires that the rate of increase of incoming number of vehicles between X and X+ΔX is equal to the rate at which vehicles flow in minus the rae at which they flow out. Thus

$$\frac{d}{dt}\int_{X}^{X+\triangle X}\rho(\hat{X},t)d\hat{X}=q(X,t)-q(X+\triangle X,t)$$

3.B

$$\lim_{\triangle X\to 0}\frac{1}{\Delta X}\int_{X}^{X+\triangle X}\frac{\partial\rho}{\partial t}(\hat{X},t)d\hat{X}=\lim_{\triangle X\to 0}\frac{q(X,t)-q(X+\triangle X,t)}{\triangle X}$$

3.C

Now applying fundementiel theored of calculs we get the balanc low in differential forms

$$\frac{\partial\rho}{\partial t}+\frac{\partial q}{\partial X}=0$$

3.D

Which tell us how the concentration ρ changes in time at each X from the follow q.

The general equation of the traffic can be usually express in the form[10,13-16]

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u(\rho)}{\partial X}=0$$

3.E

This is to be solved.

4.Solution of TRAFFIC FLOW DIFFERENTIAL EQUATION——

The partial differential equation, which was formulated to mathematically model traffic flow is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial X}(\rho u(\rho))=0$$

4.A

are equivalently

$$\frac{\partial \rho}{\partial t} + u(\rho)\frac{\partial \rho}{\partial X} + \rho \frac{\partial u(\rho)}{\partial X} = 0$$

4.B

One possible initial condition is prescribe the initial traffic density

$$\rho(X,0) = f(X)$$

It is well know to the traffic flow theory that vehicular velocity varies inversely with concentration of vehicles. So we can take $u = f(\rho)$ $=u(P(X,t))=u(P(X(t),t))=f(X,t)$ as a consequence of (3.E), we get

$$\frac{\partial \rho}{\partial t} + (u + \rho u')\frac{\partial \rho}{\partial X} = 0$$

4.C

Consider the following equation of motion which express the acceleration of the traffic stream at a given place and time as

$$\frac{du}{dt} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial X}$$

4.D

(4.D) can be generated as

$$\frac{du}{dt} = -c^2 \rho^n \frac{\partial \rho}{\partial X}$$

4.E

Take $u = f(X,t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial X}\frac{dX}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt}$$

4.F

Sub is in (4.D), we get

$$\frac{\partial u}{\partial X}u + \frac{\partial u}{\partial t} + c^2 \rho^n \frac{\partial \rho}{\partial X} = 0$$

4.G

We say that, from (4.F)

$$\frac{du}{dt} = u'\frac{\partial \rho}{\partial t}$$

4.H

Using (4.H) and (4.F), we see that (4.G) became

$$\frac{\partial \rho}{\partial t}+\left[u+\frac{c^{2}\rho^{n}}{u'}\right]\frac{\partial \rho}{\partial X}=0$$

4.I

This is generalized equation function of motion. The solution of (4.C) and (4.I) is
obtained be equaling the quanlifies within the brackets

$$(u`)^{2}=c^{2}\rho^{n-1}$$

4.J

$$\Rightarrow u'=-c\rho^{\frac{n-1}{2}}, \qquad c>0$$

4.K

To solution (4.K) for $n=-1$ and obtained

$$u=c\ln(\frac{\rho_{j}}{\rho}), \qquad c>0$$

4.L

The solution of (4.K) for $n>-1$ is

$$u=\frac{-2c}{n+1}\rho^{\frac{n+1}{2}}+c_{1,} \qquad n>-1, c>0$$

4.M

Where the constant of integration is to be evaluated by the BC.

This $c_{1}=\frac{2c}{n+1}\rho_{j}^{(\frac{n+1}{2})}, \qquad c>0,$

$$u=\frac{-2c}{n+1}\rho^{\frac{n+1}{2}}+C_{1}, \qquad c>0, n>-1,$$

Conclusion(Remark): In this module introduces fundamental balance idea necessary to derive kinematical conservation equation for traffic flow given to illustrate the complexities of the model (and the physical situation), characteristics of first-order partial differential equation are derived and used from first principles. The modeling ideas are the main emphasis of this module.

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