Numerical Solution of volterra Integral Equations with Delay Using Block Methods

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Abstract:

In this paper the Block methods which include: method of two and three blocks will be applied to Volterra integral equations with delay. The values of the involved integrals in each method are evaluated numerically using quadrature formula. Moreover, programs for each method are written in MATLAB language. A comparison between two types has been made depending on the least squares errors.

Key words: Block method, Volterra integral equation with delay.

1. Introduction:

In the last thirty years there has been a great deal of work in the field of differential equations with modified argument. These equations arise in a wide variety of scientific and technical applications, including the modeling of problems from the natural and social sciences such as physics, biological sciences and economics [7].

A special class is represented by the differential equations with affine modification of the argument which can be delay differential equation (DDE) or differential equations with linear modification of the argument. Many results concerning these equations are given in the papers [1]-[4].

These equations are equivalent to the following integral equation

$$h(x)f(x) = g(x) + \lambda \int_a^{b(x)} k(x,y) f(y-\tau)dy \qquad (1)$$

Where h, g and k are given continuous functions, λ *is a scalar parameter (we will take* λ *equal to one), and f(x) is unknown function to be determined.*

Equation (1) is called Volterra integral equation with delay when b(x)=x and it is called fredholm integral equation with delay when b(x)=b ,where b is constant, moreover it is called of the first kind if h(x)=0 and of the second kind if h(x)=1,also if g(x)=0 the equation (1) is called homogenous and called nonhomogenous if g(x)$\neq$ *0 [6].*

In this paper we consider the Block method applied to the Volterra integral equation with constant delay $\tau \succ 0$

$$f(x) = g(x) + \int_a^x k(x,y) f(y-\tau)dy \qquad (2)$$

and

$$f(x) = \Phi(x) \quad for \quad x \in [a-\tau, a) \qquad (3)$$

with given continuous function Φ.

2. Block Method:

A Block method is essentially an extrapolation procedure which has advantage of being self starting and produces a block of values at a time. Linze [5],describes two block methods and uses these methods to solve Volterra integral equation of the second kind.

In this paper this method has been used for the first time to solve Volterra integral equations of the second kind with delay ,in which a block of two and three values are produced at each stage and the values of the involved integrals are obtained using the quadrature formula.

2.1 Method of two Blocks:

Applying equation (2) with $x = x_{2n+1} = x_0 + (2n+1)h$*, and* $x = x_{2n+2} = x_0 + (2n+2)h$ *, where* $x_0 = a$ *to get:-*

$$f_{2n+1} = g_{2n+1} + \int_{x_0}^{x_{2n}} k(x_{2n+1},y) f(y-\tau)dy + \int_{x_{2n}}^{x_{2n+1}} k(x_{2n+1},y) f(y-\tau)dy \qquad (4)$$

$$f_{2n+2} = g_{2n+2} + \int_{x_0}^{x_{2n}} k(x_{2n+2}, y) f(y-\tau) dy + \int_{x_{2n}}^{x_{2n+2}} k(x_{2n+2}, y) f(y-\tau) dy \quad \text{.......................}(5)$$

This technique depends on the use of two quadrature formulas. The first is Simpson1/3 rule and the formula given in [5]:

$$\int_{x_0}^{x_1} f(x) dx = \frac{h}{2}[5 f_0 + 8 f_1 - f_2] \quad \text{...}(6)$$

with $x_0 = x_{2n}$ *and* $x_1 = x_{2n+1}$ *where* $n \geq 0$*, and the second is simpson 1/3 rule ,therefore we obtain:*

$$f_{2n+1} = g_{2n+1} + \frac{h}{3}\sum_{j=0}^{2n}[w_j k(x_{2n+1}, x_j) f(x_j - \tau)] + \frac{h}{12}[5k(x_{2n+1}, x_{2n}) f(x_{2n} - \tau) + 8k(x_{2n+1}, x_{2n+1})$$
$$f(x_{2n+1} - \tau) - k(x_{2n+1}, x_{2n+2}) f(x_{2n+2} - \tau)] \quad \text{......................}(7)$$

$$f_{2n+2} = g_{2n+2} + \frac{h}{3}\sum_{j=0}^{2n+2} \overline{w}_j k(x_{2n+2}, x_j) f(x_j - \tau) \quad \text{.......... ....}(8)$$

Where :-

$w_0 = w_{2n} = 1, w_j = 3 - (-1)^j, 1 \leq j \leq 2n-1$

and

$\overline{w}_0 = \overline{w}_{2n+2} = 1, \overline{w}_j = 3 - (-1)^j, 1 \leq j \leq 2n+1$

Thus we have a pair of equations to solve for f_{2n+1} *and* f_{2n+2}. *Algorithm (Block 2) gives of the program that has been used to express this method.*

2.2 Method of Three Blocks:

Applying equation (2) with $x = x_{3n+1}, x = x_{3n+2}$ *and* $x = x_{3n+3}$ *to get :*

$$f_{3n+1} = g_{3n+1} + \int_{a}^{x_{3n}} k(x_{3n+1}, y) f(y-\tau) dy + \int_{x_{3n}}^{x_{3n+1}} k(x_{3n+1}, y) f(y-\tau) dy \quad \text{......................}(9)$$

$$f_{3n+2} = g_{3n+2} + \int_{a}^{x_{3n}} k(x_{3n+2}, y) f(y-\tau) dy + \int_{x_{3n}}^{x_{3n+2}} k(x_{3n+2}, y) f(y-\tau) dy \quad \text{......................}(10)$$

$$f_{3n+3} = g_{3n+3} + \int_{a}^{x_{3n}} k(x_{3n+3}, y) f(y-\tau)dy + \int_{x_{3n}}^{x_{3n+3}} k(x_{3n+3}, y) f(y-\tau)dy \qquad (11)$$

This method depends on the use of three quadrature formulas. The first is simpson 3/8 rule and the formulas given by (6),the second is simpson 3/8 rule and simpson 1/3 rule ,and the third is simpson 3/8 ,therefore:-

$$f_{3n+1} = g_{3n+1} + \frac{3h}{8}\sum_{j=0}^{3n} w_j k(x_{3n+1}, x_j) f(x_j - \tau) + \frac{h}{12}[5k(x_{3n+1}, x_{3n}) f(x_{3n} - \tau) +$$
$$8k(x_{3n+1}, x_{3n+1}) f(x_{3n+1} - \tau) - k(x_{3n+1}, x_{3n+2}) f(x_{3n+2} - \tau)] \qquad (12)$$

$$f_{3n+2} = g_{3n+2} + \frac{3h}{8}\sum_{j=0}^{3n} \overline{w}_j k(x_{3n+2}, x_j) f(x_j - \tau) + \frac{h}{3}[k(x_{3n+2}, x_{3n}) f(x_{3n} - \tau) +$$
$$4k(x_{3n+2}, x_{3n+1}) f(x_{3n+1} - \tau) + k(x_{3n+2}, x_{3n+2}) f(x_{3n+2} - \tau)] \qquad (13)$$

$$f_{3n+3} = g_{3n+3} + \frac{3h}{8}\sum_{j=0}^{3n+3} \overline{\overline{w}}_j k(x_{3n+3}, x_j) f(x_j - \tau) \qquad (14)$$

Where:

$$w_0 = w_{3n} = 1 \quad , w_j = \begin{cases} 2 & , if \ \frac{j}{3} = \text{integer} \\ 3 & , otherwise \end{cases}$$

$$\overline{w}_0 = \overline{w}_{3n} = 1 \quad , \overline{w}_j = \begin{cases} 2 & , if \ \frac{j}{3} = \text{integer} \\ 3 & , otherwise \end{cases}$$

and

$$\overline{\overline{w}}_0 = \overline{\overline{w}}_{3n+3} = 1 \quad , \overline{\overline{w}}_j = \begin{cases} 2 & , if \ \frac{j}{3} = \text{integer} \\ 3 & , otherwise \end{cases}$$

Thus, we have a system of three equations to solve for f_{3n+1}, f_{3n+2} *and* f_{3n+3}. *Algorithm (Block 3) gives of the program that has been used to express this method.*

3. Algorithms:

The following algorithms (Block2 and Block3) for solving Volterra integral equation with delay using the two Block method and three Block method respectively:-

3.1 The Algorithm (Block 2):

Step (1):-

a. *put h=(b-a)/n ;* $n \in N$
b. *set* $f_0 = g_0 = g(a)$

Step (2): for j=1: n-1

Calculate f_j *and* f_{j+1} *using equations (7) and (8) and use Gauss elimination procedure to solve the resulting system.*

3.2 The Algorithm (Block 3):

Step (1):

a. *put h=(b-a)/n ;* $n \in N$
b. *set* $f_0 = g_0 = g(a)$

Step (2): for j=1 : n-2

Calculate f_j, f_{j+1} *and* f_{j+2} *using equations (12),(13) and (14) with Gauss elimination procedure to solve the resulting system.*

4. Numerical Examples:

4.1 Example (1):

Consider the following Volterra integral equation with delay:

$$f(x) = \sin x + x^2 \cos(x-1) - x^2 \cos(-1) + \int_0^x x^2 f(y-1)dy$$

with $f(x) = x - \dfrac{x^3}{3!}$ *, for* $x \in [-1,0)$

Table (1) presents results from a computer program that solves this problem for which the analytical solution is $f(x) = \sin x$ over the interval [0,1] with n=10, i.e. h=0.1, and L.S.E=least square error

Table (1): Exact and numerical solution for example (1)

x	exact	Block(2)	Block(3)
0	٠.٠٠٠٠٠٠٠٠٠٠٠٠٠٠٠	٠.٠٠٠٠٠٠٠٠٠٠٠٠٠٠٠	٠.٠٠٠٠٠٠٠٠٠٠٠٠٠٠٠
0.1	٠.٠٩٩٨٣٣٤١٦٦٦٤٦٨٣	٠.٠٩٩٨٣٣٤٤٢٠٩٥٠١	٠.٠٩٩٨٣٣٤٤٢٠٩٥٠١
0.2	٠.١٩٨٦٦٩٣٣٠٧٩٥٠٦	٠.١٩٨٦٦٩٣٢٧٣١٥٢٧	٠.١٩٨٦٦٩٣٢٧٣١٥٢٧
0.3	٠.٢٩٥٥٢٠٢٠٦٦٦١٣٤	٠.٢٩٥٥٢٠٤٨٢٢٣٦٩١	٠.٢٩٥٥٢٠١٨١٣٤٠٣٢
0.4	٠.٣٨٩٤١٨٣٤٢٣٠٨٦٥	٠.٣٨٩٤١٨٣١٦٩٤٢١٦	٠.٣٨٩٤١٨٨٤٢١٣٣٨٨
0.5	٠.٤٧٩٤٢٥٥٣٨٦٠٤٢٠	٠.٤٧٩٤٢٦٤٠٥٨٥٢٧٨	٠.٤٧٩٤٢٥٤٥٢٥٩٠٩٩
0.6	٠.٥٦٤٦٤٢٤٧٣٣٩٥٠٤	٠.٥٦٤٦٤٢٣٩٧١٥٢٥٥	٠.٥٦٤٦٤٢٣٠١٦٤٤٧٤
0.7	٠.٦٤٤٢١٧٦٨٧٢٣٧٦٩	٠.٦٤٤٢١٩٥٢٤٦٠١٠٠	٠.٦٤٤٢١٩٣٩٤٦٠٤٢٦
0.8	٠.٧١٧٣٥٦٠٩٠٨٩٩٥٢	٠.٧١٧٣٥٥٩٣٤٣٥٢٥٥	٠.٧١٧٣٥٥٧٦٤٥٦٠٩٠
0.9	٠.٧٨٣٣٢٦٩٠٩٦٢٧٤٨	٠.٧٨٣٣٣٠٠٦٢٩٠٨٢٦	٠.٧٨٣٣٢٦٤٤٨١٤٣١٩
1	٠.٨٤١٤٧٠٩٨٤٨٠٧٩٠	٠.٨٤١٤٧٠٧٢٩١١٥٩٤	٠.٨٤١٤٧٤٥٧٨٩٦٣٨٤
L.S.E		١.٤٢٤٤١٤٦٩٨٦٨e-011	1.64405447683e-011

Table (2) gives the least square error for different values of n

Table (2): Least square error

Method	n=20	n=30	n=100
Block(2)	1.1412566416e-013	6.7092505037e-015	1.4716843379e-018
Block(3)	7.5956627090e-014	3.6626499081e-015	9.3453355173e-019

4.2 Example (2):

Consider the following Volterra integral equation with delay:

$$f(x) = e^{x} - xe^{x-1} + xe^{-1} + \int_{0}^{x} xf(y-1)dy$$

with $f(x) = 1 + x + \frac{x^2}{2!}$ *, for* $x \in [-1,0)$

Table (3) presents results from a computer program that solves this problem for which the analytical solution is $f(x) = e^{x}$ over the interval [0,1] with n=10, i.e. h=0.1

Table (3): Exact and numerical solution for example (2)

x	*exact*	*Block(2)*	*Block(3)*
0	١.٠٠٠٠٠٠٠٠٠٠٠٠٠٠	١.٠٠٠٠٠٠٠٠٠٠٠٠٠٠	١.٠٠٠٠٠٠٠٠٠٠٠٠٠٠
0.1	١.١٠٥١٧٠٩١٨٠٧٥٦٥	١.١٠٥١٧٠٧٥٠٨١٨٤٥	١.١٠٥١٧٠٧٥٠٨١٨٤٥
0.2	١.٢٢١٤٠٢٧٥٨١٦٠١٧	١.٢٢١٤٠٢٧٦٧١٩٩٣٥	١.٢٢١٤٠٢٧٦٧١٩٩٣٥
0.3	١.٣٤٩٨٥٨٨٠٧٥٧٦٠٠	١.٣٤٩٨٥٨٢٠٨٢٦٩٥٨	١.٣٤٩٨٥٨٨٥٥٧٢٦٠٥
0.4	١.٤٩١٨٢٤٦٩٧٦٤١٢٧	١.٤٩١٨٢٤٧٣٧٨٠٠٦١	١.٤٩١٨٢٣٨٥٨٧٤٦٩٤
0.5	١.٦٤٨٧٢١٢٧٠٧٠٠١٣	١.٦٤٨٧٢٠٠٧٣٣٠٧٢٣	١.٦٤٨٧٢١٣٨١٤٥٤٢٦
0.6	١.٨٢٢١١٨٨٠٠٣٩٠٥١	١.٨٢٢١١٨٩٠١٠٨٤١٥	١.٨٢٢١١٩٠٢٦٦٨٢١٢
0.7	٢.٠١٣٧٥٢٧٠٧٤٧٠٤٨	٢.٠١٣٧٥٠٦٩١٦٠٩٠٤	٢.٠١٣٧٥٠٨٣٨١٤٠٠٠
0.8	٢.٢٢٥٥٤٠٩٢٨٤٩٢٤٧	٢.٢٢٥٥٤١١٢٨٦٣٢٥٣	٢.٢٢٥٥٤١٢٩٦٠٩٦٤٨
0.9	٢.٤٥٩٦٠٣١١١١٥٦٩٥	٢.٤٥٩٥٩٩٩٨٦١٧٤٩٣	٢.٤٥٩٦٠٣٧١٣٧٩٩٦٦
1	٢.٧١٨٢٨١٨٢٨٤٥٩٠٥	٢.٧١٨٢٨٢١٧٩٢١٩٥٠	٢.٧١٨٢٧٨٣٨٤١٩٨٨٨
L.S.E		*1.58250255303e-011*	*1.66532286414e-011*

Table (4) gives the least square error for different values of n

Table (4): Least square error

Method	*n=20*	*n=30*	*n=100*
Block(2)	*1.2604349930e-013*	*7.4013585764e-015*	*1.6218710487e-018*
Block(3)	*8.1977489151e-014*	*4.0958846418e-015*	*1.0027955288e-018*

5. Conclusion:

Block methods are constructed to compute numerical solution to a Volterra integral equation with delay. For each type a computer program was written and several examples were solved using proposed method.

We conclude the following remarks:

- *The three block method gives better accuracy than two block method.*
- *As n (the number of knots) increases, the error term is decreased in all of the used methods.*

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الحل العددي لمعادلات فولتيرا التكاملية التباطؤية بأستخدام طرق البلوك

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المستخلص:

في هذا البحث طرق البلوك التي تتضمن طريقة البلوك الثانية والثالثة طبقت على معادلة فولتيرا التكاملية التباطؤيةقيم التكاملات النا تجة في كل طريقة حسبت عدديا بأستخدام القواعد التربيعية.

فضلا عن ذلك كتبت البرامج الخاصة بكل طريقة بأستخدام لغة ما تلابكما و تم اجراء مقارنة بين انواع البلوك بأستخدام الأخطاء التربيعية.