

Oscillations of First Order Linear Delay Differential Equations with positive and negative coefficients

*Hussain Ali Mohamad**

*Intisar Haitham Qasem**

*Zainab Ahmed Abdullah**

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Abstract:

Oscillation criteria are obtained for all solutions of the first-order linear delay differential equations with positive and negative coefficients where we established some sufficient conditions so that every solution of (1.1) oscillate. This paper generalized the results in [11]. Some examples are considered to illustrate our main results.

Key word: Delay differential equations, oscillations.

Introduction:

The study of delay differential equation with positive (or negative) coefficients has been considered an attention of many researchers all over the world for the last several years see [1]-[3],[6],[8]-[10],and few of them investigated the case with positive and negative coefficients see [4]-[5],[7]. The authors in [11] investigated the first order neutral differential equations with positive and negative coefficients with constant delays. In this paper we generalized the result in [11] where we used variable delays. Consider the linear delay differential equation with positive and negative coefficients

$$\dot{x}(t) + P(t)x(\tau(t)) - Q(t)x(\sigma(t)) = 0 \quad \dots (1)$$

Where $P, Q \in C([t_0, \infty), R^+)$, and τ, σ are continuous strictly increasing functions with

$$\lim_{t \rightarrow \infty} \tau(t) = \infty, \lim_{t \rightarrow \infty} \sigma(t) = \infty, \quad \text{and}$$

$$\tau(t) \leq \sigma(t) < t \dots (2)$$

By a solution of Eq.(1.1) we mean a function $x \in ([t_x, \infty), R)$ such that x satisfies eq.(1.1), $t_x = \max \{\tau(t), \sigma(t)\}$.

A solution of eq.(1) is said to be oscillatory if it has arbitrarily large zeros, otherwise is said to be nonoscillatory. The purpose of this paper is to obtain sufficient conditions for the oscillation of all solutions of eq. (1).

1. Some Basic Lemmas:

The following lemmas will be useful in the proof of the main results:

Lemma 1 (theorem 2.1.1 [7]) . If $q(t) < t \quad \forall t \geq t_0$ is continuous function $\lim_{t \rightarrow \infty} q(t) = \infty$ and

$$\liminf_{t \rightarrow \infty} \int_{q(t)}^t P(s) ds > \frac{1}{e} \quad \dots (1.1)$$

Then the following statements are true:

1. $\dot{x}(t) + P(t)x(q(t)) \leq 0$ has no eventually positive solutions.
2. $\dot{x}(t) + P(t)x(q(t)) \geq 0$ has no eventually negative solutions.
3. $\dot{x}(t) - P(t)x(q(t)) \geq 0$ has no eventually positive solutions.
4. $\dot{x}(t) - P(t)x(q(t)) \leq 0$ has no eventually negative solutions.

*Dept. of Mathematics, College of science for women, University of Baghdad

Lemma 2 .(lemma 1.5.4 [5]) Let $a \in (-\infty, 0), \tau \in (0, \infty), t_0 \in R$ and suppose that a function $x \in [(t_0 - \tau, \infty), R]$ satisfies the inequality $x(t) \leq a + \max_{t-\tau \leq s \leq t} x(s)$ for $t \geq t_0$. Then x cannot be a non-negative function.

Remark. We can generalize Lemma 2 by taking $\tau(t)$ to satisfy (2) and $x(t) \leq a + \max_{\tau(t) \leq s \leq t} x(s)$, $t \geq t_0$. Then x is eventually negative function for $t \geq t_0$.

The following lemma improve lemma 2.6.1 given in [5]

Lemma 3.

Assume that (2) holds.

Let $x(t)$ be an eventually positive solution of (1) and set

$$z(t) = x(t) - \int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))x(s)ds, \quad t \geq \sigma(\tau^{-1}(t_0))$$

... (1.2)

$$u(t) = x(t) - \int_{\tau(t)}^{\sigma(t)} P(\tau^{-1}(s))x(s)ds, \quad t \geq \tau(\sigma^{-1}(t_0))$$

... (1.3)

then the following statements are true.

1- if $\int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))ds \leq 1, P(t) \geq Q(\sigma^{-1}(\tau(t)))\dot{\tau}(t)$

and $\dot{\sigma}(t) \geq 1$... (1.4)

then $z(t)$ is eventually positive and non increasing function.

2- if $\int_{\tau(t)}^{\sigma(t)} P(\tau^{-1}(s))ds \leq 1, P(\tau^{-1}(\sigma(t)))\dot{\sigma}(t) \geq Q(t)$

and $\dot{\tau}(t) \leq 1$... (1.5)

then $u(t)$ is eventually positive and non-increasing function.

Proof.

Suppose that $x(t) > 0, x(\tau(t)) > 0$, and $x(\sigma(t)) > 0, t \geq t_0$

1. Differentiate (1.2) and use (1) we get

$$\dot{z}(t) = -[P(t) - Q(\sigma^{-1}(\tau(t)))\dot{\tau}(t)]x(\tau(t)) + Q(t)x(\sigma(t))(1 - \dot{\sigma}(t))$$

Using (1.4) this yields

$$\dot{z}(t) \leq -[P(t) - Q(\sigma^{-1}(\tau(t)))\dot{\tau}(t)]x(\tau(t)) \leq 0$$

... (1.6)

to show that $z(t)$ is eventually positive, suppose that $z(t) \leq 0$ since $z(t)$ is not equivalent to 0 for $t \geq t_1 \geq t_0$ then there exists $t_2 \geq t_1$ such that $z(t_2) < 0$ then $z(t) \leq z(t_2)$ for $t \geq t_2$ then (1.2) will be

$$x(t) = z(t) + \int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))x(s)ds \leq z(t_2) + \int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))x(s)ds$$

$$\leq z(t_2) + \int_{\tau(t)}^{\sigma(t)} Q(\sigma^{-1}(s))ds(\max_{\tau(t) \leq s \leq \sigma(t)} x(s))$$

$$x(t) \leq z(t_2) + \max_{\tau(t) \leq s \leq t} x(s) \quad t \geq t_2$$

Then by lemma 2 we see that $x(t) < 0$ for $t \geq t_2$, this is a contradiction.

2. Differential (1.3) and use (1) we get

$$\dot{u}(t) = [Q(t) - P(\tau^{-1}(\sigma(t)))\dot{\sigma}(t)]x(\sigma(t)) + P(t)x(\tau(t))(\dot{\tau}(t) - 1)$$

Using (1.4) this yields

$$\dot{u}(t) \leq [Q(t) - P(\tau^{-1}(\sigma(t)))\dot{\sigma}(t)]x(\sigma(t)) \leq 0$$

... (1.7)

To show that $u(t)$ eventually positive, suppose that $u(t) \leq 0, t \geq t_0$, since $u(t)$ is not equivalent to 0 for $t \geq t_1 \geq t_0$ then there exists $t_2 \geq t_1$ such that $u(t_2) < 0$ then $u(t) \leq u(t_2)$ for $t \geq t_2$ then (1.3) will be

$$x(t) = u(t) + \int_{\tau(t)}^{\sigma(t)} P(\tau^{-1}(s))x(s)ds \leq u(t_2) + \int_{\tau(t)}^{\sigma(t)} P(\tau^{-1}(s))ds(\max_{\tau(t) \leq s \leq \sigma(t)} x(s))$$

$$x(t) \leq u(t_2) + \max_{\tau(t) \leq s \leq t} x(s) \quad \text{for } t \geq t_2$$

thus by Lemma 2 we see that $x(t) < 0, t \geq t_2$.

This is a contradiction, the proof of lemma is complete. ■

2.Main results:

The next result provid a sufficient conditions for the oscillation of all solutions of eq. (1)

Theorem1.

Assume that (2),(1.4) hold and that

$$\liminf_{t \rightarrow \infty} \int_{\tau(t)}^t [P(s) - Q(\sigma^{-1}(\tau(s))) \dot{\tau}(s)] ds > \frac{1}{e} \dots (2.1)$$

Then every solution of (1) oscillates.

Proof. Assume for the sake of contradiction that equation (1) has an eventually positive solution $x(t)$, by Lemma 3 (1) it follows that $z(t)$ which is defined by (1.2) is an eventually positive and non increasing function and $z(t) \leq x(t)$ also from (1.6) we see that eventually

$$\dot{z}(t) + [P(t) - Q(\sigma^{-1}(\tau(t))) \dot{\tau}(t)] z(\tau(t)) \leq 0$$

Or

$$\dot{z}(t) + [P(t) - Q(\sigma^{-1}(\tau(t))) \dot{\tau}(t)] z(\tau(t)) \leq 0$$

but in view of (2.1) it follow from Lemma 1(1) that the last inequality cannot have an eventually positive solutions. Which is a contradiction since $z(t)$ is eventually positive function. ■

Theorem 2.

Assume that (2),(1.6) hold and that

$$\liminf_{t \rightarrow \infty} \int_{\sigma(t)}^t [P(\tau^{-1}(\sigma(s))) \dot{\sigma}(s) - Q(s)] ds > \frac{1}{e} \dots (2.2)$$

Then every solution of equation (1) oscillates.

Proof: Assume for the sake of contradiction that (1) has eventually positive solution $x(t)$. By Lemma 2 (2) it follows that $u(t)$ which is defined by (1.3) is eventually positive and monotone decreasing function and

$u(t) \leq x(t)$. Also from (1.7) we see that eventually

$$\dot{u}(t) + [P(\tau^{-1}(\sigma(t))) \dot{\sigma}(t) - Q(t)] x(\sigma(t)) \leq 0$$

$$\dot{u}(t) + [P(\tau^{-1}(\sigma(t))) \dot{\sigma}(t) - Q(t)] u(\sigma(t)) \leq 0$$

But in view of (2.2) it follow from Lemma 1(2) that the last inequality cannot eventually have a positive solution. Which is a contradiction since $u(t)$ is eventually positive function. ■

Example 1.

Consider the delay differential equation;

$$\dot{x}(t) + \frac{t^2}{(t - \frac{5\pi}{2})^2} x(t - \frac{5\pi}{2}) - \frac{2t}{(t - 2\pi)^2} x(t - 2\pi) = 0 \quad t > \frac{5\pi}{2}$$

(E1)

One can find that conditions (1.4) and (2.1) are met so according to theorem 1 every solution of equation (E1) oscillate for instance the solution $x(t) = t^2 \sin t$ is oscillatory solution. ■

Example 2.

Consider the delay differential equation;

$$\dot{x}(t) + \frac{t - 2\pi}{t^2} x(t - 2\pi) - \frac{t - \frac{3\pi}{2}}{t} x(t - \frac{3\pi}{2}) = 0 \quad t > 0$$

(E2)

One can find that conditions (1.5) and (2.2) are met so according to theorem 2 every solution of equation (E2) oscillate for instance the solution $x(t) = \frac{1}{t} \sin t$ is oscillatory solution. ■

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تذبذب المعادلات التفاضلية التباطؤية الخطية من الرتبة الأولى ذات المعاملات الموجبة والسالبة

زينب احمد عبد الله*

انتصار هيثم قاسم*

حسين علي محمد*

*جامعة بغداد, كلية العلوم للبنات, قسم الرياضيات

الخلاصة:

في هذا البحث تمت دراسة المعادلات التباطؤية الخطية ذات المعاملات الموجبة و السالبة من الرتبة الأولى . حيث تم إيجاد شروط ضرورية وكافية لضمان تذبذب كافة حلول المعادلة $x'(t) + P(t)x(\tau(t)) - Q(t)x(\sigma(t)) = 0$, وشروط كافية أخرى للحلول غير المتذبذبة كي تكون متقاربة الى الصفر, وقد أعطينا بعض الأمثلة لتوضيح هذه النتائج .