# Some results On Fuzzy Symmetric Sets and Fuzzy subgroups

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## Abstract

The aim of this paper is to introduce the concepts of fuzzy subgroups, fuzzy cosets, normal fuzzy subgroups and we define a new notion for fuzzy symmetric sets, in particular, we prove some results on them.

## Introduction

The concept of fuzzy sets and fuzzy set operations was first introduced by (zadeh, 1965). (Rosonfeld, 1971) formulated the concept of a fuzzy subgroup and showed how some basic notions of group theory should be extended in the elementary manner to develop the theory of fuzzy subgroups. Subsequently others, (Malik, 1991) and (Akgul, 1988) defined normal fuzzy subgroups, fuzzy cosets and obtained some group theoretic analogs. The purpose of this paper is to study some result on fuzzy symmetric sets, fuzzy subgroups and normal fuzzy subgroups.

## Definitions and Preliminaries

Let $G$ be a group with multiplication. The set of all fuzzy subset of $G$ is denoted by FP (G) and $\mathbb{R}$, Z denote the set of real number and integer respectively

**Definition 1: (Zadeh, 1965)**

Let $X$ be a set and $I$ the unit interval [0.1].A fuzzy set $\widetilde{A}$ in $X$ is characterized by a membership function $M_{\tilde{A}}$ which is associate with each point $x \in X$ its "grade of membership" $M_{\tilde{A}}(x) \in I$ .

**Definition 2: (Zadeh, 1965)**

Let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy sets in $X$ then

$\widetilde{A} = \widetilde{B} \Leftrightarrow M_{\tilde{A}}(x) = M_{\tilde{B}}(x)$ For all $x \in X$

$\tilde{A} \subseteq \tilde{B} \Leftrightarrow M_{\tilde{A}}(x) \leq M_{\tilde{B}}(x)$ For all $x \in X$

$\tilde{C} = \tilde{A} \bigcup \tilde{B} \Leftrightarrow M_{\tilde{C}}(x) = \max\{M_{\tilde{A}}(x), M_{\tilde{B}}(x)\}$ For all $x \in X$

$\tilde{D} = \tilde{A} \bigcap \tilde{B} \Leftrightarrow M_{\tilde{D}}(x) = \min\{M_{\tilde{A}}(x), M_{\tilde{B}}(x)\}$ For all $x \in X$

**Definition 3: (Kim, 1997)**

We define the binary operation " $\circ$ " on *FP (G)* and unary operation on *FP (G)* as follows: for all $\tilde{A}, \tilde{B} \in FP(G)$ and for all $x \in G$.

$M_{\tilde{A}\circ\tilde{B}}(x) = \sup\{\min\{M_{\tilde{A}}(y), M_{\tilde{B}}(z)\} / y, z \in G, yz = x\}$

And, we call $\tilde{A} \circ \tilde{B}$ the product of $\tilde{A}$ *and* $\tilde{B}$ for short denoted by $\tilde{A}\tilde{B}$ and $\tilde{A}^{-1}$ the inverse of $\tilde{A}$.

**Definition 4 :( Rosenfeld, 1971)**

Let $G$ be a group and let $\tilde{A}$ *and* $\tilde{B}$ be fuzzy subsets of $G$.A fuzzy subset $\tilde{A}$ is called a fuzzy subgroup of $G$ if

$$M_{\tilde{A}}(xy) \geq \min\{M_{\tilde{A}}(x), M_{\tilde{A}}(x)\} \quad \text{for all } x, y \in G$$

$$M_{\tilde{A}}(x^{-1}) \geq M_{\tilde{A}}(x) \quad \text{for all } x \in G$$

A fuzzy set $\tilde{B}$ is called a fuzzy semi subgroup in $G$ if

$M_{\tilde{B}}(xy) \geq \min\{M_{\tilde{B}}(x), M_{\tilde{B}}(x)\}$ *for all* $x, y \in G$.

**Definition 5: (Mishref, 1995)**

Let $\tilde{A}$ be a fuzzy subgroup of $G$, $x \in G$ Then the fuzzy subset $x\tilde{A}(\tilde{A}x)$ is called a left (resp. right) fuzzy coset of $\tilde{A}$ in $G$ s.t $M_{x\tilde{A}}(g) = M_{\tilde{A}}(x^{-1}g)$ (resp. $M_{\tilde{A}x}(g) = M_{\tilde{A}}(gx^{-1})$)For all $g \in G$

**Theorem 1: (Kim, 1997)**

Let $\tilde{A} \in FP(G)$, then the following are equivalent:

1. $M_{\tilde{A}}(xy) = M_{\tilde{A}}(yx)$ For all $x, y \in G$ in this case $\tilde{A}$ is called an abelian fuzzy subset of $G$.
2. $M_{\tilde{A}}(xyx^{-1}) = M_{\tilde{A}}(y)$ For all $x, y \in G$.
3. $M_{\tilde{A}}(xyx^{-1}) \geq M_{\tilde{A}}(y)$ For all $x, y \in G$.
4. $M_{\tilde{A}}(xyx^{-1}) \leq M_{\tilde{A}}(y)$ for all $x, y \in G$.

**Definition 6: (Kim, 1997)**

Let $\tilde{A}$ be fuzzy subgroup of $G$ then $\tilde{A}$ is called a normal fuzzy subgroup of $G$, if it is abelian fuzzy subset of $G$.

**Theorem 2: (Rosenfeld, 1971)**

Let $\widetilde{A}$ be a fuzzy subgroup of $G$ then $M_{\widetilde{A}}(x) = M_{\widetilde{A}}(x^{-1})$ and $M_{\widetilde{A}}(e) \geq M_{\widetilde{A}}(x)$ for all $x \in G$ where $e$ is the identity element of $G$.

**Theorem 3: (Foster, 1979)**

Let $G, H$ be two groups and $f$ a homomorphism of $G$ into $H$. Let $\tilde{A}$ be a Fuzzy subgroup in $H$ then $f^{-1}(\widetilde{A})$ is fuzzy subgroup in $G$.

**Theorem 4: (Foster, 1979)**

Let $G, H$ be groups and $f$ a homomorphism of $G$ into $H$. Let $\widetilde{A}$ be a fuzzy subgroup in $G$ then $f(\widetilde{A})$ is a fuzzy subgroup in $H$.

**Theorem 5: (Foster, 1979)**

Let $G, H$ be two groups and $\widetilde{V}$ be a normal fuzzy subgroup of $H$ If $f$ is a homomorphism from $G$ into $H$ then $f^{-1}(\widetilde{V})$ is a normal fuzzy subgroup of $G$

**Theorem 6: (Foster, 1979)**

Let $\widetilde{A}$ be normal fuzzy subgroup of $G$ and $G, H$ be two groups. Suppose $f$ is an empimorphism of $G$ onto $H$ then $f(\widetilde{A})$ is a normal fuzzy subgroup of $H$.

## Product and unary of fuzzy subset of $G$ and fuzzy subgroup

**Theorem 1:**

Let $\widetilde{A}, \widetilde{B}, \widetilde{A}_i \in FP(G), i \in I$ then the following holds.

1- $(\widetilde{A}^{-1})^{-1} = \widetilde{A}$

2- $\widetilde{A} \subseteq \widetilde{B} \Leftrightarrow \widetilde{A}^{-1} \subseteq \widetilde{B}^{-1}$

3- $(\bigcup_{i \in I} \widetilde{A}_i)^{-1} = \bigcup_{i \in I} \widetilde{A}_i^{-1}$

4- $(\bigcap_{i \in I} \widetilde{A}_i)^{-1} = \bigcap_{i \in I} \widetilde{A}_i^{-1}$

5- $(\widetilde{A}\widetilde{B})^{-1} = \widetilde{B}^{-1}\widetilde{A}^{-1}$.

6- $\widetilde{A}\widetilde{B} \neq \widetilde{\phi}$ $iff$ $\widetilde{A} \neq \widetilde{\phi}$ $and$ $\widetilde{B} \neq \widetilde{\phi}$

7- $\widetilde{A}^{-1} \neq \widetilde{\phi}$ $iff$ $\widetilde{A} \neq \widetilde{\phi}$

8- $\widetilde{A} \subseteq \widetilde{B}$ $and$ $\widetilde{C} \subseteq \widetilde{D}$ $implies$ $\widetilde{A}\widetilde{C} \subseteq \widetilde{B}\widetilde{D}$

**Proof:**

1- $M_{(\widetilde{A}^{-1})^{-1}}(x) = M_{\widetilde{A}^{-1}}(x^{-1}) = M_{\widetilde{A}}((x^{-1})^{-1}) = M_{\widetilde{A}}(x)$ for all $x \in G$ we get $(\widetilde{A}^{-1})^{-1} = \widetilde{A}$

2- $M_{\tilde{A}^{-1}}(x) = M_{\tilde{A}}(x^{-1}) \le M_{\tilde{B}}(x^{-1}) = M_{\tilde{B}^{-1}}(x)$ for all $x \in G$ hence $\tilde{A}^{-1} \subseteq \tilde{B}^{-1}$

Conversely

$M_{\tilde{A}}(x) = M_{(\tilde{A}^{-1})^{-1}}(x) = M_{\tilde{A}^{-1}}(x^{-1}) \le M_{\tilde{B}^{-1}}(x^{-1}) = M_{(\tilde{B}^{-1})^{-1}}(x) = M_{\tilde{B}}(x) \quad by(1)$ For all $x \in G$ we get $\tilde{A} \subseteq \tilde{B}$

3- $M_{(\bigcup_{i\in I}\tilde{A}_i)^{-1}}(x) = M_{(\bigcup_{i\in I}\tilde{A}_i)}(x^{-1}) = \sup\{M_{\tilde{A}_i}(x^{-1})\} = \sup\{M_{\tilde{A}^{-1}}(x)\} = M_{\bigcup_{i\in I}\tilde{A}_i^{-1}}(x)$

Hence $(\bigcup_{i\in I}\tilde{A}_i)^{-1} = \bigcup_{i\in I}\tilde{A}_i^{-1}$

4- Proof similarly as 3

5-

$$M_{(\tilde{A}\tilde{B})^{-1}}(x) = M_{\tilde{A}\tilde{B}}(x^{-1}) = \sup\{\min\{M_{\tilde{A}}(y^{-1}), M_{\tilde{B}}(z^{-1})\} / y^{-1}, z^{-1} \in G, x^{-1} = y^{-1}z^{-1}\}$$
$$= \sup\{\min\{M_{\tilde{A}^{-1}}(y), M_{\tilde{B}}(z)\} / y, z \in G, x = zy\}$$
$$= M_{\tilde{B}^{-1}\tilde{A}^{-1}}(x) \text{ For all } x \in G$$

**Theorem 2:**

If $\tilde{A}$ is a fuzzy subset on a group $G$. Then for every $x, y, g \in G$ :

1- $M_{x\tilde{A}}(g) = M_{\tilde{A}}(x^{-1}g)$.

2- $M_{\tilde{A}x}(g) = M_{\tilde{A}}(gx^{-1})$.

3- $(xy)\tilde{A} = x(y\tilde{A})$.

4- $\tilde{A}(xy) = (\tilde{A}x)y$.

**Proof:**

1. $M_{x\tilde{A}}(g) = \sup_{g=p_1p_2}\{\min\{M_x(p_1), M_{\tilde{A}}(p_2)\}\}$
$$= \sup\{\min\{M_x(x), M_{\tilde{A}}(x^{-1}g)\}\} = M_{\tilde{A}}(x^{-1}g)$$

2. $M_{\tilde{A}x}(g) = \sup_{g=p_1p_2}\{\min\{M_x(p_1), M_{\tilde{A}}(p_2)\}\}$
$$= \sup\{\min\{M_x(x), M_{\tilde{A}}(gx^{-1})\}\} = M_{\tilde{A}}(gx^{-1}).$$

3. $M_{xy\tilde{A}}(g) = \sup_{g=p_1p_2}\{\min\{M_x(p_1), M_{\tilde{A}}(p_2)\}\}$
$$= \sup\{\min\{M_{xy}(xy), M_{\tilde{A}}(y^{-1}x^{-1}g)\}\} = M_{\tilde{A}}(y^{-1}x^{-1}g) = M_{y\tilde{A}}(x^{-1}g) = M_{x(y\tilde{A})}(g)$$

Hence $(xy)\tilde{A} = x(y\tilde{A})$.

4. Similarly as 3

**Lemma 1:**

Let $G$ and $H$ be two groups and $f$ a homomorphism of $G$ into $H$ then

1- For any fuzzy subsets $\tilde{A}$ and $\tilde{B}$ of $H$, $f^{-1}(\tilde{A})f^{-1}(\tilde{B}) \subseteq f^{-1}(\tilde{A}\tilde{B})$.

2- For any fuzzy subsets $\tilde{A}$ and $\tilde{B}$ of $H$ , $f(\tilde{A})f(\tilde{B}) = f(\tilde{A}\tilde{B})$.

**Proof:**

1- $M_{f^{-1}(\tilde{A})f^{-1}(\tilde{B})}(x) = \sup_{x=x_1x_2}\{\min\{M_{f^{-1}(\tilde{A})}(x_1), M_{f^{-1}(\tilde{B})}(x_2)\}\}$

$$= \sup_{x-x_1x_2}\{\min\{M_{\tilde{A}}(f(x_1)), M_{\tilde{B}}(f(x_2))\}\}$$

$$= \sup_{x-x_1x_2}\{\min\{M_{\tilde{A}\tilde{B}}(f(x)\}\} \text{ For all } x \in G$$

$$\leq M_{f^{-1}(\tilde{A}\tilde{B})}(x) \text{ For all } x \in G$$

$$f^{-1}(\tilde{A})f^{-1}(\tilde{B}) \subseteq f^{-1}(\tilde{A}\tilde{B})$$

2- let $\tilde{A}$ and $\tilde{B}$ be subsets of $G$ and $y \in f(G)$ then

$M_{f(\tilde{A}\tilde{B})}(y) = \sup_{y=f(x)}\{\sup_{x=st}\{\min\{M_{\tilde{A}}(s), M_{\tilde{B}}(t)\}\}\}\ldots$ (a)

And

$M_{f(\tilde{A})f(\tilde{B})}(y) = \sup_{y=uv}\{\sup_{v=f(p)}\{\sup_{v=f(q)}\{\min\{M_{\tilde{A}}(p), M_{\tilde{B}}(q)\}\}\}\}\ldots$ (b)

Equality of (a) and (b) follows by putting $p = s$ and $q = t$ .

Example 1:

$G = \{-1,1\}$ Is a group with multiplication, define $\tilde{A} = \{(1,0.3), (-1,0.4)\}$, $\tilde{B} = \{(1,0.2), (-1,0.7)\}$. $\tilde{A}, \tilde{B}$ Are fuzzy subsets of $G$. To find $\tilde{A}\tilde{B}$.

$M_{\tilde{A}\tilde{B}}(x) = \sup_{x=x_1x_2}\{\min\{M_{\tilde{A}}(x_1), M_{\tilde{B}}(x_2)\}\}$ For all $x \in G$

$$M_{\tilde{A}\tilde{B}}(1) = \sup_{\substack{1=1.1 \\ 1=-1.-1}}\{\min\{M_{\tilde{A}}(\mp 1), M_{\tilde{B}}(\mp 1)\}\}$$

$$= \sup\{0.2, 0.4\} = 0.4$$

$$M_{\tilde{A}\tilde{B}}(-1) = \sup_{\substack{-1=1.-1 \\ -1=-1.1}}\{\min\{M_{\tilde{A}}(\mp 1), M_{\tilde{B}}(\pm 1)\}\}$$

$$= \sup\{0.3, 0.2\} = 0.3$$

$$\tilde{A}\tilde{B} = \{(1,0.4), (-1,0.3)\}$$

Example 2:

$G = \{-1,1\}$ Is group with multiplication. Define $\tilde{A} = \{(1,0.3), (-1,0.4)\}$, $\tilde{B} = \{(1,0.2), (-1,0.7)\}$. $\tilde{A}, \tilde{B}$ Is fuzzy subset of $G$.To find $\tilde{A}\tilde{B}$ see example 3.1Now $f(x) = x^2$ which implies that $f(xy) = (xy)^2 = x^2y^2 = f(x)f(y)$. $f$ Is homomorphism.

$M_{f^{-1}(\tilde{A}\tilde{B})}(x) = M_{\tilde{A}\tilde{B}}(f(x)) = M_{\tilde{A}\tilde{B}}(x^2)$. $M_{f^{-1}(\tilde{A}\tilde{B})}(\mp 1) = M_{\tilde{A}\tilde{B}}(1) = 0.4$

$f^{-1}(\tilde{A}) = \{(\mp 1, 0.3)\}$, $f^{-1}(\tilde{B}) = \{(\mp 1, 0.2)\}$

$f^{-1}(\tilde{A})f^{-1}(\tilde{B}) = \{(\mp 1, 0.2)\}$, $f^{-1}(\tilde{A})f^{-1}(\tilde{B}) \neq f^{-1}(\tilde{A}\tilde{B})$

This means that equality of lemma 3.1 part 1 is not true.

Example 3:

$G = \{-1, 1\}$ Is group with multiplication. Define $\tilde{A} = \{(1, 0.3), (-1, 0.4)\}$, $\tilde{B} = \{(1, 0.2), (-1, 0.7)\}$. $\tilde{A}, \tilde{B}$ Are fuzzy subsets of $G$ to find $\tilde{A}\tilde{B}$.

$M_{\tilde{A}\tilde{B}}(x) = \sup_{x = x_1 x_2}\{\min\{M_{\tilde{A}}(x_1), M_{\tilde{B}}(x_2)\}\}\ \forall x \in G$

$M_{\tilde{A}\tilde{B}}(1) = \sup_{\substack{1 = 1.1 \\ 1 = -1.-1}}\{\min\{M_{\tilde{A}}(\mp 1), M_{\tilde{B}}(\mp 1)\}\}$

$= \sup\{0.2, 0.4\} = 0.4$

$M_{\tilde{A}\tilde{B}}(-1) = \sup_{\substack{-1 = 1.-1 \\ -1 = -1.1}}\{\min\{M_{\tilde{A}}(\mp 1), M_{\tilde{B}}(\pm 1)\}\}$

$= \sup\{0.3, 0.2\} = 0.3$

$\tilde{A}\tilde{B} = \{(1, 0.4), (-1, 0.3)\}$

Now $f(x) = x^2$ which implies that $f(xy) = (xy)^2 = x^2 y^2 = f(x)f(y)$

Hence $f$ is homomorphism.

$M_{f(\tilde{A}\tilde{B})}(y) = \sup\{M_{\tilde{A}\tilde{B}}(x), x \in G\ f(x) = y\}\ \forall y \in G$

$M_{f(\tilde{A}\tilde{B})}(1) = \sup\{0.4, 0.3\}$ $f(\tilde{A}\tilde{B}) = \{(1, 0.4)\}$

$f(\tilde{A}) = \{(1, 0.4)\}$, $f(\tilde{B}) = \{(1, 0.7)\}$.

$M_{f(\tilde{A})f(\tilde{B})}(1) = \sup\{\min\{M_{f(\tilde{A})}(\mp 1), M_{f(\tilde{B})}(\mp 1)\}\} = \sup\{0.4, 0\} = 0.4$

Hence $f(\tilde{A})f(\tilde{B}) = f(\tilde{A}\tilde{B})$.

**Theorem 3:**

Every fuzzy subgroup is a fuzzy semisubgroup

**Proof**: Obvious

**Remark 1:**

The converse of theorem (3.3) is not true in general, and the following example shows this fact.

Example: ($\mathbb{R}$-{0},.) is a group , define $\tilde{A}: \mathbb{R}\text{-}\{0\} \to [0,1]$ as

$$M_{\tilde{A}}(x) = \begin{cases} 1 & if\ x \in Z - \{0\} \\ o & if\ x \notin Z - \{0\} \end{cases}$$

Clearly $\tilde{A}$ is a fuzzy semi subgroup of $\mathbb{R}$- {0}, but not fuzzy subgroup Of $\mathbb{R}$- {0} if we take $x = 2$ $x^{-1} = \frac{1}{2}$ , $M_{\tilde{A}}(x^{-1}) \geq M_{\tilde{A}}(x)$

We get $0 \geq 1$ which is a contradiction.

**Proposition 1:**

Let $\widetilde{A}$ be a fuzzy subgroup of G. if there exists a sequence $\{x_n\}$ in G such that $\lim M_{\tilde{A}}(x_n) = 1 as\, n \to \infty$ then $M_{\tilde{A}}(e) = 1$

**Proof:** by theorem 2.2, we have $M_{\tilde{A}}(e) \geq M_{\tilde{A}}(x)$ for all $x \in G$ . Thus, we get $M_{\tilde{A}}(e) \geq M_{\tilde{A}}(x_n)$ for every positive integer. Consider $1 \geq M_{\tilde{A}}(e) \geq \lim M_{\tilde{A}}(x_n) = 1$ as $n \to \infty$ we get $1 \geq M_{\tilde{A}}(e) \geq 1$.Therefore $M_{\tilde{A}}(e) = 1$**.**

**Proposition 2:**

If $\widetilde{A}$ is a fuzzy subgroup then for all $\alpha \in \mathrm{Im}(\widetilde{A})$ , $\widetilde{A}_\alpha = \{x \in G / M_{\tilde{A}}(x) \geq \alpha\}$ is a subgroup of G.

**Proof:** Let $x, y \in \widetilde{A}_\alpha$ to show that $xy^{-1} \in \widetilde{A}_\alpha$

$M_{\tilde{A}}(xy^{-1}) \geq \min\{M_{\tilde{A}}(x), M_{\tilde{A}}(y)\} = \min\{\alpha, \alpha\} = \alpha$, we get $xy^{-1} \in \widetilde{A}_\alpha$**.**

**Proposition 3:**

If $\widetilde{A}$ is a fuzzy subgroup, then for all $\alpha \in [0,1]$ , $\widetilde{A}_\alpha = \{x \in G / M_{\tilde{A}}(x) > \alpha\}$ is a subgroup of G.

**Proof:** Obvious

**Proposition 4:**

If $\widetilde{A}$ is a fuzzy subgroup of G then $\ker \widetilde{A} = \{x \in G : M_{\tilde{A}}(x) = 1\}$ is a subgroup.

**Proof:** Let $a, b \in \ker \widetilde{A}$ to prove $ab^{-1} \in \ker \widetilde{A}$
$M_{\tilde{A}}(ab^{-1}) \geq \min\{M_{\tilde{A}}(a), M_{\tilde{A}}(b)\} = 1$. Since $\widetilde{A}$ is a fuzzy subgroup we get $ab^{-1} \in \ker \widetilde{A}$**,** *hence* $\ker \widetilde{A}$ *is a* subgroup.

**Proposition 5:**

Each subgroup of G is a level subgroup of a fuzzy subgroup of G

**Proof:** Let Y be a subgroup of G and let $\widetilde{A}$ be an fuzzy set on G define by

$$M_{\tilde{A}}(x) = \begin{cases} \alpha & if\ x \in Y \\ 0 & if\ x \notin Y \end{cases}$$

Where $\alpha \in [0,1]$.It is clear that $\{x \in G / M_{\tilde{A}}(x) \geq \theta\} = Y$

Let $x, y \in G$. We have the following cases

**1-** If $x \in Y\ and\ y \notin Y$ then $M_{\tilde{A}}(x) = \alpha\ and\ M_{\tilde{A}}(y) = 0$ thus

$M_{\tilde{A}}(xy^{-1}) \geq 0 = \min\{\alpha, 0\} = \min\{M_{\tilde{A}}(x), M_{\tilde{A}}(y)\}$

**2-** If $x \notin Y$ *and* $y \in Y$ then $M_{\tilde{A}}(y) = \alpha$ *and* $M_{\tilde{A}}(x) = 0$ thus
$M_{\tilde{A}}(xy^{-1}) \geq 0 = \min\{\alpha, 0\} = \min\{M_{\tilde{A}}(x), M_{\tilde{A}}(y)\}$
**3-** *if* $x \notin Y$ *and* $y \notin Y$ then $M_{\tilde{A}}(x) = 0$ *and* $M_{\tilde{A}}(y) = 0$ thus
$M_{\tilde{A}}(xy^{-1}) \geq 0 = \min\{M_{\tilde{A}}(x), M_{\tilde{A}}(y)\}$
**4-**If *if* $x \in Y$ *and* $y \in Y$ then $M_{\tilde{A}}(x) = \alpha$ *and* $M_{\tilde{A}}(y) = \alpha$ thus
$M_{\tilde{A}}(xy^{-1}) = \alpha = \min\{\alpha, \alpha\} = \min\{M_{\tilde{A}}(x), M_{\tilde{A}}(y)\}$.

**Proposition 6:**

Let $Y$ be a subset of $G$ and $\tilde{A}$ be a fuzzy subset on $G$ which was defined in proposition (3.5). If $\tilde{A}$ is a fuzzy subgroup of $G$ then $Y$ is also a subgroup of $G$.

**Proof:** Let $\tilde{A}$ be a fuzzy subgroup of $G$ and $x, y \in Y$ then $M_{\tilde{A}}(x) = \alpha = M_{\tilde{A}}(y)$, $M_{\tilde{A}}(xy^{-1}) \geq \min\{M_{\tilde{A}}(x), M_{\tilde{A}}(y)\} = \min\{\alpha, \alpha\} = \alpha$
$xy^{-1} \in Y$ .

**Proposition 7:**

Let $\tilde{N}$ be a fuzzy subset of $G$ defined as

$$M_{\tilde{N}}(x) = \begin{cases} \alpha & if\ x \in N \\ \sigma & if\ x \notin N \end{cases}$$

For all $\alpha, \sigma \in [0,1]$ *with* $\alpha \geq \sigma$ , then $\tilde{N}$ is a fuzzy subgroup of $G$ if and only if $\tilde{N}$ is a subgroup of $G$, moreover, in this case $\tilde{A}_* = \tilde{N}$

**Proof:** Let $\tilde{N}$ be a fuzzy subgroup and $x, y \in G$ be such that $x, y \in \tilde{N}$ then
$M_{\tilde{N}}(xy^{-1}) \geq \min\{M_{\tilde{N}}(x), M_{\tilde{N}}(y)\} = \min\{\alpha, \alpha\} = \alpha$ , Hence $xy^{-1} \in \tilde{N}$
Conversely,

Suppose $\tilde{N}$ is a subgroup of G. Let $x, y \in G$
If $x, y \in \tilde{N}$ then $xy^{-1} \in \tilde{N}$, $M_{\tilde{N}}(xy^{-1}) \geq \alpha = \min\{M_{\tilde{N}}(x), M_{\tilde{N}}(y)\}$
If $x \notin \tilde{N}$ *or* $y \notin \tilde{N}$ *then* $M_{\tilde{N}}(xy^{-1}) \geq \sigma = \min\{M_{\tilde{N}}(x), M_{\tilde{N}}(y)\}$
Hence $\tilde{N}$ is a fuzzy subgroup. Moreover, we have
$\tilde{A}_* = \{x \in G / M_{\tilde{N}}(x) = M_{\tilde{N}}(e)\} = \{x \in G / M_{\tilde{N}}(x) = \alpha\} = \tilde{N}$ .

**Proposition 8**:

Let $G$ be a group. Then two level subgroup $\tilde{A}_{\alpha_1}, \tilde{A}_{\alpha_2}$ where ( $\alpha_1 < \alpha_2$ ) of $\tilde{A}$ are equal if and only if there is no $x \in G$ such that $\alpha_1 \leq M_{\tilde{A}}(x) < \alpha_2$ where
$\tilde{A}_\alpha = \{x \in G / M_{\tilde{A}}(x) \geq \alpha\}$

**Proof:** Let $\widetilde{A}_{\alpha_1} = \widetilde{A}_{\alpha_2}$ where( $\alpha_1 < \alpha_2$) and there exist $x \in G$ s.t $\alpha_1 \le M_{\tilde{A}}(x) < \alpha_2$.Then $\widetilde{A}_{\alpha_2}$ is proper subset of $\widetilde{A}_{\alpha_1}$ which is a contradiction
Conversely, assume that there is no $x \in G$ s.t $\alpha_1 \le M_{\tilde{A}}(x) < \alpha_2$ since $\alpha_1 < \alpha_2$
Then $\widetilde{A}_{\alpha_2} \subseteq \widetilde{A}_{\alpha_1}$ .If $x \in \widetilde{A}_{\alpha_1}$ then $M_{\tilde{A}}(x) \ge \alpha_1$ by hypotheses we get
$M_{\tilde{A}}(x) \ge \alpha_2$, therefore $x \in \widetilde{A}_{\alpha_2}$ then $\widetilde{A}_{\alpha_1} \subseteq \widetilde{A}_{\alpha_2}$
Hence $\widetilde{A}_{\alpha_1} = \widetilde{A}_{\alpha_2}$ **.**

**Theorem 4:**
Let $\widetilde{H}_1, \widetilde{H}_2$ be two fuzzy subgroups of $G$ then $\widetilde{H}_1 \cup \widetilde{H}_2$ is a fuzzy subgroup if and only if $\widetilde{H}_1 \subseteq \widetilde{H}_2$ or $\widetilde{H}_2 \subseteq \widetilde{H}_1$ .

**Proof:** $M_{\tilde{H}_1 \cup \tilde{H}_2}(xy^{-1}) = \sup\{M_{\tilde{H}_1}(xy^{-1}), M_{\tilde{H}_2}(xy^{-1})\}$
$= M_{\tilde{H}_1}(xy^{-1})$ Or $M_{\tilde{H}_2}(xy^{-1})$
Hence $\widetilde{H}_1 \cup \widetilde{H}_2 = \widetilde{H}_1$ or $\widetilde{H}_1 \cup \widetilde{H}_2 = \widetilde{H}_2$ .
Conversely, to prove $\widetilde{H}_1 \subseteq \widetilde{H}_2$ or $\widetilde{H}_2 \subseteq \widetilde{H}_1$ . Let $\widetilde{H}_1 \not\subset \widetilde{H}_2$ or $\widetilde{H}_2 \not\subset \widetilde{H}_1$ .
$M_{\tilde{H}_1}(x) > M_{\tilde{H}_2}(x)$ & $M_{\tilde{H}_2}(x) > M_{\tilde{H}_1}(x)$
We get $M_{\tilde{H}_1}(x) > M_{\tilde{H}_2}(x) > M_{\tilde{H}_2}(x) > M_{\tilde{H}_1}(x)$ $M_{\tilde{H}_1}(x) > M_{\tilde{H}_1}(x)$. Which contradiction because $\widetilde{H}_1 = \widetilde{H}_2$.

## Fuzzy Symmetric, Fuzzy Coset and Normal Fuzzy Subgroup

**Definition 1:**
A fuzzy subset $\widetilde{A}$ of a group $G$ is called symmetric if $(\widetilde{A})^{-1} = \widetilde{A}$

Example 1:
$G = \{1,-1\}$ Is group with multiplication. Define $\widetilde{A} = \{(1,0.3),(-1,0.4)\}$, then $\widetilde{A}^{-1} = \widetilde{A}$ .Hence $\widetilde{A}$ is symmetric.

**Theorem 1:**
Every fuzzy subgroup $\widetilde{A}$ of $G$ is fuzzy symmetric set.

**Proof:** To prove $(\widetilde{A})^{-1} = \widetilde{A}$ we have to prove $M_{\tilde{A}}(x) = M_{(\tilde{A})^{-1}}(x)$ for all $x \in G$. $M_{\tilde{A}}(x) = M_{\tilde{A}}(x^{-1}) = M_{(\tilde{A})^{-1}}(x)$ For all $x \in G$ (since $\widetilde{A}$ is fuzzy subgroup)**.** Hence $(\widetilde{A})^{-1} = \widetilde{A}$ .

**Remark 1:**
The converse of theorem (4.1) is not true in general see example 4.1

**Lemma 1:**
Every symmetric fuzzy semi subgroup is a fuzzy subgroup.
**Proof:** Obvious
**Lemma 2:**
Let $G$ and $H$ be two groups and $f$ a homomorphism of $G$ into $H$ then
1- For any fuzzy symmetric subset $\tilde{A}$ of $G$, $f(\tilde{A})$ is fuzzy symmetric in $H$.
2- For any fuzzy symmetric subset $\tilde{B}$ of $H$, $f^{-1}(\tilde{B})$ is fuzzy symmetric in $G$.
**Proof:**
1- $M_{f(\tilde{A})^{-1}}(y) = M_{f(\tilde{A})}(y^{-1}) = \sup\{M_{\tilde{A}}(z), z \in G, f(z) = y^{-1}\}$
$= \sup\{M_{\tilde{A}^{-1}}(z), z \in G, f(z) = y^{-1}\}$ Since $\tilde{A} = \tilde{A}^{-1}$
$= \sup\{M_{\tilde{A}}(z^{-1}), z^{-1} \in G, f(z^{-1}) = y\} = M_{\tilde{A}}(y)\ \forall y \in H$
Therefore $(f(\tilde{A}))^{-1} = f(\tilde{A})$ .
2-
$M_{(f(\tilde{B}))^{-1}}(x) = M_{f^{-1}(\tilde{B})}(x^{-1}) = M_{\tilde{B}}(f(x^{-1})) = M_{\tilde{B}^{-1}}(f(x^{-1})) = M_{\tilde{B}}(f(x^{-1}))^{-1}$.
$= M_{\tilde{B}}(f(x)) = M_{f^{-1}(\tilde{B})}(x) \quad \forall x \in G$
Hence $f^{-1}((\tilde{B}))^{-1} = f^{-1}(\tilde{B})$.
**Lemma 3:**
Let $\tilde{A}$ be a fuzzy subgroup of $G$ Then $xG^t_{\tilde{A}} = G^t_{x\tilde{A}}$ for every $x \in G$, $t \in [0,1]$ where $G^t_{\tilde{A}} = \{y \in G, M_{\tilde{A}}(y) \geq t\}$.
**Proof:** Now $G^t_{x\tilde{A}} = \{g \in G, M_{x\tilde{A}}(g) \geq t\}$ by definition 2.5
$G^t_{x\tilde{A}} = \{g \in G, M_{\tilde{A}}(x^{-1}g) \geq t\ \forall g \in G\}$ .
Let $b = x^{-1}g$ since G is a group $xx^{-1} = e$ we get $xb = g$
$G^t_{x\tilde{A}} = \{xb \in G, M_{\tilde{A}}(b) \geq t\} = xG^t_{\tilde{A}}$ .
Example 2:
Let $G = \{\mp 1, \mp i\}$ be the group with respect to multiplication. Define $\tilde{A} : G \to [0,1]$ as follows

$$M_{\tilde{A}}(x) = \begin{cases} \frac{1}{2} & if \quad x = -1 \\ 1 & if \quad x = 1 \\ \frac{1}{4} & if \quad x = i, -i \end{cases}$$

The coset $i\tilde{A}$ is calculated as follows

$$M_{i\tilde{A}}(x)=\begin{cases}\frac{1}{4} & if \quad x=1,-1\\ 1 & if \quad x=i\\ \frac{1}{2} & if \quad x=-i\end{cases}$$

It is clear that $i\tilde{A}$ is not a fuzzy subgroup because $M_{i\tilde{A}}(-i.-i)=M_{i\tilde{A}}(i^2)=M_{i\tilde{A}}(1)\geq\min\{M_{i\tilde{A}}(-i),M_{i\tilde{A}}(-i)\}$this implies that $\frac{1}{4}\geq\frac{1}{2}$, this is contradiction .

**Theorem 2:**

Every fuzzy subgroup $\tilde{A}$ of $G$ and $t\in\operatorname{Im}(\tilde{A})$ then $G^1_{\tilde{A}}$ is a subgroup of $G^t_{\tilde{A}}$

**Proof:** Let $x,y\in G^1_{\tilde{A}}$ we have to prove $xy^{-1}\in G^1_{\tilde{A}}$ .now

$M_{\tilde{A}}(xy^{-1})\geq\min\{M_{\tilde{A}}(x),M_{\tilde{A}}(y)\}=1$ , we get $xy^{-1}\in G^1_{\tilde{A}}$ .

Hence $xy^{-1}\in G^1_{\tilde{A}}$ is subgroup of $G^t_{\tilde{A}}$ .

**Corollary 1:**

If $f:G\to G$ is a mapping such that $f(x)=axa^{-1},a\in G$ and $\tilde{A}$ is a fuzzy subgroup of $G$ then $a\tilde{A}a^{-1}$ is a fuzzy subgroup of $G$

**Proof:** Now $f(x)=axa^{-1}$ is a homomorphism $f(xy)=axya^{-1}=axa^{-1}aya^{-1}=f(x)f(y)$, $f(\tilde{A})$ is a fuzzy subgroup of $G$ by theorem 2.4 which implies that $a\tilde{A}a^{-1}$ is a fuzzy subgroup of $G$.

**Theorem 3:**

Every fuzzy subgroup of an abelian group is a normal fuzzy subgroup.

**Proof:** Let $\tilde{A}$ be a fuzzy subgroup of abelian group $G$.

Now $M_{\tilde{A}}(xyx^{-1})=M_{\tilde{A}}(xx^{-1}y)=M_{\tilde{A}}(ey)=M_{\tilde{A}}(y)$ by definition (2.6) we get $\tilde{A}$ is a normal fuzzy subgroup of $G$.

**Theorem 4**:

If $\tilde{A}$ is a normal fuzzy subgroup of $G$ then $Ker\tilde{A}=\{x\in G:M_{\tilde{A}}(x)=1\}$ is normal fuzzy subgroup of $G$.

**Proof:** Let $a\in G$ and $y\in Ker\tilde{A}$. We have to prove $aya^{-1}\in Ker\tilde{A}$.

$M_{\tilde{A}}(aya^{-1})=M_{\tilde{A}}(y)=1$ Since $\tilde{A}$ is a normal fuzzy subgroup.

We get $aya^{-1}\in Ker\tilde{A}$.

Hence $Ker\tilde{A}$ is a normal subgroup of $G$.

**Theorem 5**:

Each subgroup of an abelian group $G$ is a level normal subgroup of a normal fuzzy subgroup of $G$.

**Proof:** Its clear that $Y$ is normal subgroup of $G$ and $\tilde{A}$ be a fuzzy set on $G$ defined by

$$M_{\tilde{A}}(x) = \begin{cases} \alpha & if \quad x \in Y \\ 0 & if \quad x \notin Y \end{cases} \qquad \dots (1)$$

Where $\alpha \in [0,1]$ it is clear that $\{x : x \in G, M_{\tilde{A}}(x) \geq \alpha\} = Y$ by (proposition 3.5) $\tilde{A}$ is a fuzzy subgroup of $G$ . To prove $\tilde{A}$ is a normal fuzzy subgroup of $G$. Let $x, y \in G$ .We have two cases

(1) If $y \in Y$ then $xyx^{-1} \in Y$ implies $M_{\tilde{A}}(xyx^{-1}) = M_{\tilde{A}}(y)$ by (1)

(2) If $y \notin Y$ then $M_{\tilde{A}}(y) = 0$ then $M_{\tilde{A}}(yxx^{-1}) = M_{\tilde{A}}(ye) = M_{\tilde{A}}(y) = 0$ we get $M_{\tilde{A}}(xyx^{-1}) = M_{\tilde{A}}(y)$. Hence $\tilde{A}$ is a normal fuzzy subgroup of $G$ .

**Theorem 6:**

If $\tilde{A}$ is a normal fuzzy subgroup then $\tilde{A}_{\alpha} = \{x : x \in G, M_{\tilde{A}}(x) \geq \alpha\}$ Is normal subgroup of $G$ for all $\alpha \in [0,1]$.

**Proof:** Obvious

**Corollary 2:**

If $\tilde{A}$ is a normal fuzzy subgroup then $\tilde{A}_{\alpha} = \{x : x \in G, M_{\tilde{A}}(x) \geq \alpha\}$ is a normal subgroup of $G$ for all $\alpha \in \mathrm{Im}(\tilde{A})$

**Proof:** Obvious.

**Corollary 3:**

If $f : G \to G$ is a mapping such that $f(x) = axa^{-1}$ and $\tilde{A}$ is a normal fuzzy subgroup, then $a\tilde{A}a^{-1}$ is a normal fuzzy subgroup of $G$ .

**Proof:** Obvious

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# بعض النتائج على المجموعة المتناظرة الضبابية و الزمر الجزئية الضبابية

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## الخلاصة

الهدف من البحث هو لبيان افكار الزمر الجزئية الضبابية و الزمر الجزئية الضبابية الاعتيادية ووضحنا تعريف جديد لمصطلع المجموعة المتناظرة الضبابية ،و بشكل خاص تم دراسة بعض النتائج حولها.