Exponential Stability And Uniform Exponential Stability For Nonlinear Systems

Shaheed Nasser Huseen
Computer Department
College of science
Thi-qar University

Abstract:

new bounds for Lyapunov function and we In this paper, we give a new hypothesis depending on Lyapunov function for state a establishing two types of stability which are exponential stability and uniform exponential stability for nonlinear systems.

الأستقرارية الأسية و الأستقرارية ألأسية المنتظمة للأنظمة اللاخطية

شهيد ناصر حسين
جامعة ذي قار / كلية العلوم
قسم علوم الحاسبات

الخلاصة

قمنا في هذا البحث بإعطاء قيود جديده لدالة ليابانوف كذلك وضعنا فرضيات تعتمد على دالة ليابانوف للحصول على نوعين من الاستقرارية هما الأستقرارية الأسية و الأستقرارية الأسية المنتظمة للأنظمة اللاخطية.

1.Introduction :

The investigation of stability analysis of nonlinear systems using the method has produced avast body of important results and second Lyapunov have been widly stadied[1,2,4,5,11]
The second method of Lyapunov attempts to give information on the stability of equilibrium state of linear and nonlinear systems without any prior knowledge of their solutions.
The second method of Lyapunov is based on a generalization of the idea that is the system has an asymptotically stable equilibrium state,and then the stord energy of the system displaced within the domain of attraction decays with increasing time until it finaly assumes its minimum value at the equilibrium state.The method consists of determination of a functions (energy) function called the Lyapunov function ,which is more general than that of energy and is more widely applicable [6].
The linear systems where the asymptotic stability implies the exponential stability,the exponential stability for nonlinear time –varying systems ,in general,may not be easily verified .Only,a few investigations have with exponential stability conditions for nonlinear time –varying systems[9,10]. Moreover,the problem of Lyapunov characterization of exponential stability of nonlinear time-varying systems with Lyapunov functions has remained open.
Important engineering processes involve time-varying linear and nonlinear models[3,8].
Asystem that is exponentially stable has the property that the state converges exponentially to zero state irrespective of the initial state We furthermore see that if a time invariant linear system is asymptotically stable the convergence of the state to the zero state is exponential.

2.preliminaries:

The following notation will be used this paper:
is the n-dimensional Euclidean vector space, R^+ **is the set of all non-** R^n
negative real numbers , $\|x\|$ **is the Euclidean norm of a vector** $x \in R^n$
Consider the nonlinear system described by the time-varying differential equation:

$$x'(t) = f(x(t),t), t \geq 0$$
$$x(t_0) = x_0, t_0 \geq 0 \quad (1)$$

Where $x(t) \in R^n$, $f(x,t): R^n \times R^+ \to R^n$ **is a given nonlinear function satisfying** $f(0,t) = 0$ **for all** $t \in R^+$**.we shall assume that conditions are imposed on system (1) such that the existence of its solutions is guaranteed.**

Definition 2.1:[7]

A function $V(x,t): R^n \times R^+ \to R^n$ with continuous first partial derivatives satisfying the following conditions :

a. $V(x,t) > 0$ **for all** $x \neq 0$ **in** Ω **and all t**

$V(0,t) = 0$ **for all t**

b. $V'(x,t) < 0$ **for all** $x \neq 0$ **in** Ω **and all t**

$V'(0,t) = 0$ **for all t**

Where Ω **is the region , is called the Lyapunov function.**

Definition 2.2[10]

The zero solution of system (1) is exponentially stable if any solution $x(x_0,t)$ **of (1) satisfies:**

$$\|x(x_0,t)\| \le \beta(\|x_0\|, t_0)\exp(-\delta(t-t_0)\forall t \ge t_0$$

Where $\beta(h,t): R^+ \times R^+ \to R^+$ **is a non-negative function increasing in h**$\in R^+$**, and** δ **is a positive constant .**

Remark 2.3:[10]

If the function $\beta(.)$ **in the definition (2.2) dose not depend on** t_0 **, the zero solution is called uniformly exponentially stable.**

Definition 2.4:

A function $V(x,t): R^n \times R^+ \to R$ is called a Lyapunov-Like function for (1) if $V(x,t)$ **is a continuously differentiable in** $t \in R^+$ **and in** $x \in R^n$**, and there exist positive numbers** $\lambda_1, \lambda_2, \lambda_3, p, q, r, \delta$ **and non-negative number k, such that:**

$$\lambda_1 \sqrt[p]{\|x\|} \le V(x,t) \le \lambda_2 \sqrt[q]{\|x\|}, \forall (x,t) \in R^n \times R^+ \qquad (2.4.1)$$

$$V'(x,t) \le -\lambda_3 \sqrt[r]{\|x\|} - k\exp(-\delta t), \forall t \ge 0, x \in R^n / \{0\} \qquad (2.4.2)$$

3. Main result:

The following theorems for (uniformly exponential, exponential) stability of a nonlinear system is discussed theoretically with new conditions are assumed.

Theorem 3.1:

The system (1) is uniformly exponentially stable if it admits a Lyapunov-Like function and the following two conditions hold:

a. $\exists \gamma > 0 : V(x,t) - V^{q/r}(x,t) \le \gamma \exp(-\delta t), where \delta > \lambda_3 / \lambda_2^{q/r}$

b. $m\gamma \ge k > 0, where\ m = \lambda_3 / \lambda_2^{q/r}$ **for all** $(x,t) \in R^n \times R^+$

proof:

consider $Q(x,t) = V(x,t)\exp(m(t - t_0))$ **where** $V(x,t)$ **is Lyapunov-Like function**

)) $Q'(x,t) = V'(x,t)\exp(m(t - t_0)) + mV(x,t)\exp(m(t - t_0$

From (2.4.2) we obtian:

$$Q'(x,t) \le [-\lambda_3 \|x\|^{1/r} - k\exp(-\delta t)]\exp(m(t - t_o)) + mV(x,t)\exp(m(t - t_o))$$

From the right hand of (2.4.1) we have:

which implies $-\|x\|^{1/r} \le -[V(x,t)/\lambda_2]^{q/r}\ \|x\|^{1/q} \ge V(x,t)/\lambda_2$

So that:

$$Q'(x,t) \le [-V^{q/r}(x,t)\lambda_3 / \lambda_2^{r/q} - k\exp(-\delta t)]\exp(m(t - t_0)) + mV(x,t)\exp(m(t - t_0))$$

From the condition (a) we obtain:

$$Q'(x,t) \le m\gamma\exp(-\delta t)\exp(m(t - t_0)) - k\exp(-\delta t)\exp(m(t - t_0))$$

Multiplying the right hand by $\exp(\delta t_0) \ge 1$ **, we obtain:**

$$Q'(x,t) \le m\gamma\exp(-\delta(t - t_0))\exp(m(t - t_0)) - k\exp(-\delta(t - t_0))\exp(m(t - t_0))$$

$$Q'(x,t) \le (m\gamma - k)\exp((m - \delta)(t - t_0))$$

Hence:

$$Q(x,t) - Q(x_0,t_0) \le \int_{t_0}^{t} (m\gamma - k)\exp((m - \delta)(s - t_0))ds$$

$$Q(x,t) \le Q(x_0,t_0) + (m\gamma - k)/(m - \delta)[\exp((m - \delta)(t - t_0)) - 1]$$

$$Q(x,t) \le Q(x_0,t_0) - (m\gamma - k)/(\delta - m)\exp((m - \delta)(t - t_0)) + (m\gamma - k)/(\delta - m)$$

Since $(m\gamma - k)/(\delta - m)\exp((m - \delta)(t - t_0)) \ge 0, then:$

$$Q(x,t) \le Q(x_0,t_0) + (m\gamma - k)/(\delta - m)$$

Since $Q(x_0,t_0) = V(x_0,t_0)$ **, then:**

$$Q(x,t) \le V(x_0,t_0) + (m\gamma - k)/(\delta - m)$$

From the right hand of (2.4.1) we obtain:

$$Q(x,t) \le \lambda_2 \|x_0\|^{1/q} + (m\gamma - k)/(\delta - m)$$

Setting : $\lambda_2 \|x_0\|^{1/q} + (m\gamma - k)/(\delta - m) = \beta(\|x_0\|) \ge 0$**, then:**

$$Q(x,t) \le \beta(\|x_0\|)$$

From the left hand of (2.4.1):

$$\|x\| \le [V(x,t)/\lambda_1]^p$$

$$\|x\| \le [Q(x,t)/\lambda_1\exp(m(t - t_0))]^p \le [\beta(\|x_0\|\exp((-m(t - t_0))/\lambda_1]^p$$

$$\|x\| \le [\beta(\|x_0\|)/\lambda_1]^p\exp(-mp(t - t_0))$$

Therefore the system(1) is uniformly exponentially stable.

Corollary 3.2:

The system (1) is uniformly exponentially stable if it admits a Lyapunov-Like function and satisfies the condition (a) of theorem(3.1),and k=0.

Proof:

From theorem (3.1), when k=0, we have :

$\beta(\|x_0\|) = \lambda_2 \|x_0\|^{1/q} + m\gamma/(\delta - m) \geq 0$

And:

$\|x\| \leq [\beta(\|x_0\|)/\lambda_1]^p \exp(-mp(t-t_0))$

Hence the system (1) is uniformly exponentially stable.

Lemma 3.3:

In theorem (3.1) if we chose $Q(x,t) = V(x,t)\exp(mt)$ **then the system (1) is exponentially stable.**

Proof:

$Q'(x,t) = V'(x,t)\exp(mt) + mV(x,t)\exp(mt)$

From (2.4.2) we obtain:

$Q'(x,t) \leq [-\lambda_3 \|x\|^{1/r} - k\exp(-\delta t)]\exp(mt) + mV(x,t)\exp(mt)$

From the right hand of (2.4.1) we have:

$-\|x\|^{1/r} \leq -[V(x,t)/\lambda_2]^{q/r}$

Hence:

$Q'(x,t) \leq m[V(x,t) - V^{q/r}(x,t)]\exp(mt) - k\exp((m-\delta)t)$

From the condition (a) we obtain:

$Q'(x,t) \leq (m\gamma - k)\exp((m-\delta)t)$

Hence:

$$Q(x,t) - Q(x_0,t_0) \leq \int_{t_0}^{t} (m\gamma - k)\exp((m-\delta)s)ds$$

$$Q(x,t) \leq Q(x_0,t_0) + (m\gamma - k)/(\delta - m)\exp((m-\delta)t_0) - (m\gamma - k)/(\delta - m)\exp((m-\delta)t)$$

Then:

$Q(x,t) \leq Q(x_0,t_0) + (m\gamma - k)/(\delta - m)\exp((m-\delta)t_0)$

Since $Q(x_0,t_0) = V(x_0,t_0)\exp(mt_0)$ **then:**

$Q(x,t) \leq V(x_0,t_0)\exp(mt_0) + (m\gamma - k)/(\delta - m)\exp((m-\delta)t_0)$

$Q(x,t) \leq \lambda_2 \|x_0\|^{1/q} \exp(mt_0) + (m\gamma - k)/(\delta - m)\exp((m-\delta)t_0)$

Setting $\lambda_2 \|x_0\|^{1/q} \exp(mt_0) + (m\gamma - k)/(\delta - m)\exp((m-\delta)t_0) = \beta(\|x_0\|, t_0) \geq 0$

Hence:

$Q(x,t) \leq \beta(\|x_0\|, t_0)$

Now, from the left hand of (2.4.1),we have:

$\|x\|^{1/p} \le V(x,t)/\lambda_1$

Then:

$\|x\| \le [\beta(\|x_0\|,t_0)/\lambda_1]^p \exp(-mpt)$

Hence the system (1) is exponentially stable.

Corollary 3.4:

The system (1) is exponentially stable if it is admits a Lyapunov-Like function and satisfies the condition (a) of theorem (1) and k=0.

Let us chose a new bounds for Lyapunov-Like function as the following:

$\lambda_1\|x\|^p \le V(x,t) \le \lambda_2\|x\|^q$

$V'(x,t) \le -\lambda_3\|x\|^r - k\exp(\delta t)\exp(-m(t-t_0))$

Theorem 3.5:

The system(1) with above Lyapunov-Like function is (exponentially, uniformly exponentially) stable if the following condition hold:

such that $V(x,t) - V^{r/q}(x,t) \le \gamma\exp(-\delta t)$ **, where** $\delta > m = \lambda_3/\lambda_2^{r/q}\ \exists\gamma \ge 0$

Proof:

Let $Q(x,t) = V(x,t)\exp(m(t-t_0))$**, where** $V(x,t)$ **is the above Lyapunov-Like function**

$Q'(x,t) = V'(x,t)\exp(m(t-t_0)) + mV(x,t)\exp(m(t-t_0))$

$\le [-\lambda_3\|x\|^r - k\exp(\delta t)\exp(-m(t-t_0))]\exp(m(t-t_0)) + mV(x,t)\exp(m(t-t_0))$

$\le -\lambda_3\|x\|^r\exp(m(t-t_0)) - k\exp(\delta t) + mV(x,t)\exp(m(t-t_0))$

Since:

then $-\|x\|^r \le -[V(x,t)/\lambda_2]^{r/q}$ $\|x\| \ge [V(x,t)/\lambda_2]^{1/q}$

Hence:

$Q(x,t) \le -V^{r/q}(x,t)(\lambda_3/\lambda_2^{r/q})\exp(m(t-t_0)) - k\exp(\delta t) + mV(x,t)\exp(m(t-t_0))$

$\le m[V(x,t) - V^{r/q}(x,t)]\exp(m(t-t_0)) - k\exp(\delta t)$

From the condition of theorem we obtain:

$Q(x,t) \le m\gamma\exp(-\delta t)\exp(m(t-t_0)) - k\exp(\delta t)$

Multiplying the right hand by $\exp(\delta t_0) \ge 1$ **we obtain:**

$Q(x,t) \le m\gamma\exp((m-\delta)(t-t_0)) - k\exp(\delta(t+t_0))$

$$Q(x,t) - Q(x_0,t_0) \le \int_{t_0}^{t} m\gamma\exp((m-\delta)(s-t_0))ds - k\int_{t_0}^{t}\exp(\delta(s+t_0))ds$$

$$\leq (m\gamma/(m-\delta))[\exp((m-\delta)(t-t_0))-1]-(k/\delta)[\exp(\delta(t+t_0))-\exp(2\delta t_0)]$$
$$\leq (m\gamma/(\delta-m))-(m\gamma/(\delta-m))\exp((m-\delta)(t-t_0))+(k/\delta)\exp(2\delta t_0)-(k/\delta)\exp(\delta(t+t_0))$$
$$\leq (m\gamma/(\delta-m))+(k/\delta)\exp(2\delta t_0)$$

Hence:

$$Q(x,t)\leq Q(x_0,t_0)+(m\gamma/(\delta-m))+(k/\delta)\exp(2\delta t_0)$$
$$\leq V(x_0,t_0)+(m\gamma/(\delta-m))+(k/\delta)\exp(2\delta t_0)$$

If k>0 then:

$$Q(x,t)\leq \lambda_2\|x_0\|^q+(m\gamma/(\delta-m))+(k/\delta)\exp(2\delta t_0)$$

Setting: $\lambda_2\|x_0\|^q+(m\gamma/(\delta-m))+(k/\delta)\exp(2\delta t_0)=\beta(\|x_0\|,t_0)\geq 0$

Since: $\lambda_1\|x\|^p\leq V(x,t)$ **then:**

$$\|x\|\leq [V(x,t)/\lambda_1]^{1/p}$$
$$\|x\|\leq [Q(x,t)/\lambda_1\exp(m(t-t_0))]^{1/p}$$
$$\|x\|\leq [\beta(\|x_0\|,t_0)/\lambda_1]^{1/p}\exp((-m/p)(t-t_0))$$

Hence the system (1) is exponentially stable.

If k=0 then:

$$Q(x,t)\leq \lambda_2\|x_0\|^q+m\gamma/(\delta-m)$$

Setting: $\lambda_2\|x_0\|^q+m\gamma/(\delta-m)=\beta(\|x_0\|)\geq 0$

Hence:

$$\|x\|\leq [\beta(\|x_0\|)/\lambda_1]^{1/p}\exp((-m/p)(t-t_0))$$

Therefore the system (1) is uniformlly exponentially stable.

Remark 3.6:

1. **in theorem (3.5) if the second bound of Lyapunov-Like function as the following:**

$$V'(x,t)\leq -\lambda_3\|x\|^r-k\exp(-\delta t)\exp(-m(t-t_0))$$

Then the system(1) is uniformlly exponentially stable.

2. **in theorem (3.5) if we chose** $Q(x,t)=V(x,t)\exp(mt)$**, then the system(1) is exponentially stable.**
3.

Remark 3.7:

It should be noted that, the choice of Lyapunov-Like function and $Q(x,t)$ **of effects the type of stability of the system.**

REFERENCES

[1] Bellman B., Stability Theory of Differential Equations. Mac Graw-Hill,1959.

[2] Burton T.A. and Makay G.,Marachkov Type Stability Results for Functional Differential Equations . Elect.J.Qual.Theory of Diff.Eqts.,1(1998),1-16.

[3] Ha I.J., and Chong S., Design of a Clos Guidance Law Via Feedback Linearization , IEEE Transactions on Aerospace and Electronic system, 28,51-63,1992.

[4] Katko J. and Jaki P., Stability Via The Lyapunov Function With a Discontinuous Derivative.J.Math.Anal.Appl.,4(1999),73-85.

[5] Lakshmikantham V., Leela S. and Martynyuk A., Stability Analysis of Non-linear System. Marcel Dekker , New York ,1989.

[6] Ogata K., Modren Control Engineering , Second Edition, Prentice-Hall of India,1996.

[7] Ogata K., State Space Analysis of Control System , Prentice-Hall, Englewood,Cliffs,USA, 1967.

[8] Song Y.D., and Middleton,R.H., Dealing With The Time Varying Parameter Problem of Robot Manipulators Performing Path Tracking Tasks, IEEE Transactions on Outomatic Control,37,1597-1601,1992.

[9] Sun Y.J., Hsieh J.G.and Hsieh Y.C., Exponential Stability Criteria for Uncertain Retarded Systems With Multiple Delays.J.Math.Anal.Appl.,20(1996),430-446.

[10] Sun Y.J.,Lien C.H.and Hsieh J.G., Global Exponential Stabilization for a Class of Uncertain Nonlinear Systems With Control Constraints.IEEE Trans.Aut.Contr.,(1998) ,67-70.

[11] Zabczyk J., Mathematical Control Theory : An Introduction,Birkhauser,Berlin,1992.