A Generalization of Baskakov Operators of Summation-Integral-Phillips Type Form

***Ali J. Mohammad*[1] *and Amal K. Hassan*[2]**

[1] ***University of Basrah, College of Education, Dept of Mathematics, Basrah, IRAQ.***
Email:alijasmoh@gmail.com

[2] ***University of Basrah, College of Science, Dept of Mathematics, Basrah, IRAQ.***
Email: amal_ math 2012@yahoo.com

Abstract:

In this paper, we introduce and study a generalization form of the summation-integral-Phillips Baskakov type operators. We prove that the operators are converge to the function being approximated. Also, we discuss a Voronovaskaja-type asymptotic formula and obtain an error estimate in terms of the modulus of continuity for these operators.

KEY WORDS: Linear positive operators, Simultaneous approximation, Voronovaskaja-type asymptotic formula, Degree of approximation, Modulus of continuity.

1. Introduction

Suppose that $C[0,\infty)$ denotes the space of all continuous real-valued functions on the interval $[0,\infty)$.The subspace $C_\alpha[0,\infty)$ of the space $C[0,\infty)$ is defined as

$$C_\alpha[0,\infty) = \{ f \in C[0,\infty) : f(t) = O\left((1+t)^\alpha\right), \text{for some } \alpha > 0\}.$$

The space $C_\alpha[0,\infty)$ is normed by the norm:

$$\| f \|_{C_\alpha} = \sup_{t\in[0,\infty)} | f(t)| (1+t)^{-\alpha}$$

For $f \in C_\alpha[0,\infty)$, Vijay Gupta and others (2006) defined and studied the following summation-integral Baskakov type operators:

$$M_n(f;x)=(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)\,f(t)\,dt\text{, where}$$

$$p_{n,k}(x)=\binom{n+k-1}{k}x^k(1+x)^{-n-k}\,;\ x\in[0,\infty). \qquad (1.1)$$

After that, Agrawal and Thamer (1998) defined and studied the following summation-integral-Phillips Baskakov type operators

$$K_n(f;x)=(n-1)\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k-1}(t)f(t)dt+f(0)\,p_{n,0}(x).$$

Many operators have been built and studied by using Baskakov weight functions mixed with other weight functions. Here we refer to Vijay Gupta and others(2003,2006),Linsen Xie and Xiaoping Zhang (2007) .

Here, we defined and study the following generalization form of summation-integral-Phillips Baskakov typ operators :

$$S_{n,\nu}(f;x)=(n-1)\sum_{k=\nu}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k-\nu}(t)f(t)dt+f(0)\sum_{k=0}^{\nu-1}p_{n,k}(x). \qquad (1.2)$$

Where $\sum_{k=0}^{\nu-1}f(0)p_{n,k}(x)=0$ whenever $\nu=0$.

Clear tha $S_{n,0}(f;x)=M_n(f;x)$, $S_{n,1}(f;x)=K_n(f;x)$ and all results ofband P.N.Agrawal and Thamer(1998) can be getting them here by putting $\nu=0$ and $\nu=1$ respectively.

Sometime, we write the operators $S_{n,\nu}(f;x)$ as $S_{n,\nu}(f;x)=\int_0^{\infty}W_{n,\nu}(t,x)f(t)dt$, where

$W_{n,\nu}(t,x)$ is called the kernel of the operators $S_{n,\nu}(f,x)$ and is defined as:

$$W_{n,\nu}(t;x)=(n-1)\sum_{k=\nu}^{\infty}p_{n,k}(x)p_{n,k-\nu}(t)+\sum_{k=0}^{\nu-1}\delta(t)p_{n,k}(x),$$

where $\delta(t)$ being the Dirac-delta function.

In this paper we study some properties of the operators $S_{n,\nu}(f,x)$, i.e. the convergence theorem, Voronovaskaja type asymptotic formula and the estimate of the degree of approximation for smooth functions.

Throughout this paper, we assume that C is a positive constant not necessarily the same at all occurrences and $[\beta]$ denotes the value of the integer part of β.

2. Preliminary Results

For $f\in C_\alpha[0,\infty)$ the Baskakov operators is defined by A.Sahai and G.Prasad (1985) as :

$$L_n(f;x)=\sum_{k=0}^{\infty}p_{n,k}(x)\,f\left(\frac{k}{n}\right),\ x\in[0,\infty)$$ and for

$m\in N^0$ (the set of nonnegative integers), the $m-$th order moment of the Baskakov perators is defined as $\mu_{n,m}(x)=\sum_{k=0}^{\infty}p_{n,k}(x)\left(\frac{k}{n}-x\right)^m$.

Lemma 2.1. A.Sahai and G.Prasad (1985) proved that :the function $\mu_{n,m}(x)$ defined above has the following properties: $\mu_{n,0}=1$, $\mu_{n,1}=0$ and

$n\mu_{n,m+1}(x)=x(1+x)[\mu'_{n,m}(x)+m\mu_{n,m-1}(x)],\ m\geq 1.$

Further, we have the following consequences of $\mu_{n,m}(x)$:

(i) $\mu_{n,m}(x)$ is a polynomial in x of degree at most m;

(ii) For every $x\in[0,\infty)$, $\mu_{n,m}(x)=O\left(n^{-[(m+1)/2]}\right)$.

From the above lemma, we get:

$$\sum_{k=\nu}^{\infty}p_{n,k}(x)(k-nx)^{2j}=n^{2j}\left(\mu_{n,2j}(x)-\sum_{k=0}^{\nu-1}p_{n,k}(x)\left(k/n-x\right)^{2j}\right)$$

$$=n^{2j}\left\{O(n^{-j})+O(n^{-s})\right\}\quad\text{(for any } s>0)$$

$$=O(n^{j})\ \text{(if } s\geq j). \qquad (2.1)$$

For $m\in N^0$, the $m-$th order moment $T_{n,m,\nu}(x)$ for the operators (1.2) is defined as:

$$T_{n,m,\nu}(x)=S_{n,\nu}\left((t-x)^m;x\right)=(n-1)\sum_{k=\nu}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k-\nu}(t)(t-x)^m\,dt+\sum_{k=0}^{\nu-1}p_{n,k}(x)(-x)^m.$$

Lemma2.2. For the function $T_{n,m,\nu}(x)$, we have $T_{n,0}(x)=1$,

$$T_{n,1,\nu}(x)=\frac{2x+1-\nu}{n-2}-\frac{1}{n-2}\left[\sum_{k=0}^{\nu-1}(k+1-\nu)\,p_{n,k}(x)\right],$$

$$T_{n,2,\nu}(x)=\frac{2nx^2+2nx+\nu^2-3\nu+2+6x+6x^2-6\nu x}{(n-2)(n-3)}$$

$$+\frac{1}{(n-2)(n-3)}\left(-\nu^2+3\nu-2+2nx-2n\nu x-6x+6\nu x\right)\sum_{k=0}^{\nu-1}p_{n,k}(x)$$

$$+\frac{1}{(n-2)(n-3)}(2\nu-3+2nx-6x)\sum_{k=0}^{\nu-1}kp_{n,k}(x)-\frac{1}{(n-2)(n-3)}\sum_{k=0}^{\nu-1}k^2p_{n,k}(x).$$

Also, we have the following recurrence relation for $T_{n,m,\nu}(x)$ whenever $n>m+2$,

$$(n-m-2)T_{n,m+1,\nu}(x)=x(1+x)T'_{n,m,\nu}(x)+\left((2x+1)m+2x+1-\nu\right)T_{n,m,\nu}(x)$$

$$+2mx(1+x)T_{n,m-1,\nu}(x)+(-x)^m\sum_{k=0}^{\nu-1}(\nu-1-k)p_{n,k}(x). \qquad (2.2)$$

And for every $x\in[0,\infty)$, $T_{n,m,\nu}(x)=O\left(n^{-[(m+1)/2]}\right)$.

Proof: By direct computation, we have $T_{n,0,\nu}(x)=1$,

$$T_{n,1,\nu}(x)=\frac{2x+1-\nu}{n-2}-\frac{1}{n-2}\left[\sum_{k=0}^{\nu-1}(k+1-\nu)\,p_{n,k}(x)\right]$$

$$T_{n,2,\nu}(x)=\frac{2nx^2+2nx+\nu^2-3\nu+2+6x+6x^2-6\nu x}{(n-2)(n-3)}$$

$$+\frac{1}{(n-2)(n-3)}\left(-\nu^2+3\nu-2+2nx-2n\nu x-6x+6\nu x\right)\sum_{k=0}^{\nu-1}p_{n,k}(x)$$

$$+\frac{1}{(n-2)(n-3)}(2\nu-3+2nx-6x)\sum_{k=0}^{\nu-1}kp_{n,k}(x)-\frac{1}{(n-2)(n-3)}\sum_{k=0}^{\nu-1}k^2p_{n,k}(x).$$

Next we prove (2.2). For $x=0$ it clearly holds.

For $x\in(0,\infty)$, we have :

$$T'_{n,m,\nu}(x)=(n-1)\sum_{k=\nu}^{\infty}p'_{n,k}(x)\int_0^\infty p_{n,k-\nu}(t)(t-x)^m\,dt-mp_{n,k}(x)\int_0^\infty p_{n,k-\nu}(t)(t-x)^{m-1}$$

$$+\sum_{k=0}^{\nu-1}p'_{n,k}(x)(-x)^m-m(-x)^{m-1}p_{n,k}(x).$$

Using the relations $x(1+x)p'_{n,k}(x)=(k-nx)\,p_{n,k}(x)$, we get:

$$x(1+x)T'_{n,m,\nu}(x)=(n-1)\sum_{k=\nu}^{\infty}(k-nx)\,p_{n,k}(x)\int_0^\infty p_{n,k-\nu}(t)(t-x)^m\,dt$$

$$-mx(1+x)(n-1)\sum_{k=\nu}^{\infty}p_{n,k}(x)\int_0^\infty p_{n,k-\nu}(t)(t-x)^{m-1}dt$$

$$+\sum_{k=0}^{\nu-1}(k-nx)p_{n,k}(x)(-x)^m$$

$$-m\sum_{k=0}^{\nu-1}(-x)^{m-1}p_{n,k}(x)$$

$$= (n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^{\infty} t(1+t)\, p'_{n,k-\nu}(t)(t-x)^m\, dt + (n-1)\nu \sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-\nu}(t)(t-x)^m$$

$$+ nT_{n,m+1}(x) - mx(1+x)T_{n,m-1}(x) + (-x)^m \sum_{k=0}^{\nu-1} kp_{n,k}(x).$$ By using

the identity $t(1+t) = (t-x)^2 + (1+2x)(t-x) + x(1+x)$, we have

$$x(1+x)T'_{n,m,\nu}(x) = (n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^{\infty} p'_{n,k-\nu}(t)(t-x)^{m+2}\, dt$$

$$+ (1+2x)(n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^{\infty} p'_{n,k-\nu}(t)(t-x)^{m+1}\, dt$$

$$+ x(1+x)(n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^{\infty} p'_{n,k-\nu}(t)(t-x)^m\, dt$$

$$+ (n-1)\nu \sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-\nu}(t)(t-x)^m\, dt$$

$$+ nT_{n,m+1,\nu}(x) - mx(1+x)T_{n,m-1,\nu}(x) + (-x)^m \sum_{k=1}^{\nu-1} kp_{n,k}(x)$$

Integrating by parts, we get

$$x(1+x)T'_{n,m,\nu}(x) = (n-m-2)\, T_{n,m+1,\nu}(x) - [(1+2x)m + 2x + 1 - \nu]T_{n,m,\nu}(x) - 2mx(1+x)T_{n,m-1,\nu}(x)$$

$$+ (-x)^m \sum_{k=0}^{\nu-1} (k+1-\nu)p_{n,k}(x)$$

from which (2.2) is immediate.

From the values of $T_{n,0,\nu}(x)$ and $T_{n,1,\nu}(x)$, Using the induction on m, the recurrence relation above and the fact that $\sum_{k=0}^{\nu-1} (k+1-\nu)p_{n,k}(x) = o(1)$ *as* $n \to \infty$ we can easily prove that $x \in [0,\infty), T_{n,m,\nu}(x) = O\left(n^{-[(m+1)/2]}\right)$. ■

From the above lemma, we have

$$(n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-\nu}(t)(t-x)^{2\gamma}\, dt = T_{n,2\gamma,\nu} - \sum_{k=0}^{\nu-1} p_{n,k}(x)(-x)^{2\gamma}$$

$$= O(n^{-\gamma}) + O(n^{-s}) \text{ (for any } s > 0\text{)}$$

$$= O(n^{\gamma}) \text{ (if } s \geq \gamma\text{).} \qquad (2.3)$$

Lemma 2.3. Let δ and γ be any two positive real numbers and $[a,b] \subset (0,\infty)$. Then, for any $s > 0$, we have

$$\left\| \int_{|t-x|\geq\delta} W_{n,\nu}(t,x)t^{\gamma}\,dt \right\|_{C[a,b]} = O(n^{-s}).$$

Making use of Schwarz inequality for integration , summation and (2.3), the proof of the lemma easily follows.

Lemma 2.4. Sinha,Agrawal and Vijay Gupta (1991) show that: There exist polynomials $Q_{i,j,r}(x)$ independentof n and k suchthat

$$x^r(1+x)^r\ D^r\ (p_{n,k}(x)) = \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i(k-nx)^j Q_{i,j,r}(x)\, p_{n,k}(x)\text{, where } D=\frac{d}{dx}.$$

3. Main Results

Firstly, we show that the derivative $S_{n,\nu}^{(r)}(f:x)$ is an approximation process for $f^{(r)}(x),\ r=1,\ 2,\ldots.$

Theorem 3.1. If $r\in N$, $f\in C_{\alpha}[0,\infty)$ for some $\alpha>0$ and $f^{(r)}$ exists at a point $x\in(0,\infty)$, then

$$\lim_{n\to\infty} S_{n,\nu}^{(r)}(f;x)= f^{(r)}(x). \tag{3.1}$$

Further, if $f^{(r)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty)$, $\eta>0$, then (3.1) holds uniformly in $[a,b]$.

Proof: By Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^i+\varepsilon(t,x)(t-x)^r,$$

where, $\varepsilon(t,x)\to 0$ *as* $t\to x$.

Hence

$$S_{n,\nu}^{(r)}(f;x)=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\int_0^{\infty}W_{n,\nu}^{(r)}(t,x)(t-x)^i\,dt+\int_0^{\infty}W_{n,\nu}^{(r)}(t,x)\varepsilon(t,x)(t-x)^r\,dt$$

$$:= I_1+I_2.$$

Now, using Lemma 2.2 we get that $S_{n,\nu}(t^m;x)$ is a polynomial in x of degree exactly m, for all $m\in N^0$. Further , we can write it as:

$$S_{n,\nu}(t^m;x)=\frac{(n+m-1)!(n-m-2)!}{(n-1)!(n-2)!}x^m$$

$$+\frac{(n+m-2)!\,(n-m-2)!}{(n-1)!(n-2)!}m(m-\nu)\ x^{m-1}+o(1). \tag{3.2}$$

Therefore,

$$I_1=\sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^\infty W_n^{(r)}(t,x)\,t^j\,dt$$

$$+\frac{f^{(r)}(x)}{r!}\left(\frac{(n+r-1)!(n-r-2)!}{(n-1)!(n-2)!}r!\right)$$

$$+f^{(r)}(x)\left(\frac{(n+r-1)!(n-r-2)!}{(n-1)!(n-2)!}\right)\to f^{(r)}(x) \qquad \text{as } n\to\infty$$

Next, making use of Lemma 2.4 we have

$$|I_2|\le \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r}(n-1)\sum_{k=\nu}^{\infty}p_{n,k}(x)|k-nx|^j\int_0^\infty p_{n,k-\nu}(t)|\varepsilon(t,x)||t-x|^r\,dt$$

$$+\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r}\sum_{k=0}^{\nu-1}p_{n,k}(x)\left|(k-nx)^j\right||\varepsilon(0,x)|x^r$$

$$:=I_3+I_4.$$

Since $\varepsilon(t,x)\to 0$ as $t\to x$, then for a given $\varepsilon>0$, there exists a $\delta>0$ such that $|\varepsilon(t,x)|<\varepsilon$, whenever $0<|t-x|<\delta$. For $|t-x|\ge\delta$, there exists a constant $C>0$ such that $\left|\varepsilon(t,x)(t-x)^r\right|\le C|t-x|^\gamma$.

Now, since $\sup\limits_{\substack{2i+j\le r\\ i,j\ge 0}}\dfrac{|Q_{i,j,r}(x)|}{x^r}:=M(x)=C\ \forall\, x\in(0,\infty)$ then,

$$I_3\le C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(n-1)\sum_{k-\nu}^{\infty}p_{n,k}(x)|k-nx|^j\left(\varepsilon\int_{|t-x|<\delta}p_{n,k-\nu}(t)|t-x|^r\,dt+\int_{|t-x|\ge\delta}p_{n,k-\nu}(t)|t-x|^\gamma\,dt\right)$$

$$:=I_5+I_6.$$

Now, applying Schwartz inequality for integration, summation, (2.1) and (2.3) we conclude

$$I_5\le\varepsilon C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(n-1)\sum_{k=\nu}^{\infty}p_{n,k}(x)|k-nx|^j\left(\int_0^\infty p_{n,k-\nu}(t)\,dt\right)^{1/2}\left(\int_0^\infty p_{n,k-\nu}(t)(t-x)^{2r}\,dt\right)^{1/2}$$

(since $\int_0^\infty p_{n,k-\nu}(t)\ dt=\dfrac{1}{n-1}$)

$$\le\varepsilon C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left(\sum_{k=\nu}^{\infty}p_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left((n-1)\sum_{k=\nu}^{\infty}p_{n,k}(x)\int_0^\infty p_{n,k-\nu}(t)(t-x)^{2r}\,dt\right)^{1/2}$$

$$\leq \varepsilon\, C\, O(n^{-r/2}) \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O(n^{j/2}) = \varepsilon\, O(1).$$

Again, using Schwarz inequality for integration and then for summation, in view of (2.1) and Lemma 2.3, we have

$$I_6 \leq C \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i (n-1) \sum_{k=\nu}^{\infty} p_{n,k}(x)|k-nx|^j \int_{|t-x|\geq\delta} p_{n,k-\nu}(t)|t-x|^{\gamma}\, dt$$

$$\leq C \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i (n-1) \sum_{k=\nu}^{\infty} p_{n,k}(x)|k-nx|^j \left(\int_{|t-x|\geq\delta} p_{n,k-\nu}(t)\right)^{1/2} \left(\int_{|t-x|\geq\delta} p_{n,k-\nu}(t)(t-x)^{2\gamma}\, dt\right)^{1/2}$$

$$\leq C \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i \left(\sum_{k=\nu}^{\infty} p_{n,k}(x)(k-nx)^{2j}\right)^{1/2} \left((n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k-\nu}(t)(t-x)^{2\gamma}\right)^{1/2}$$

$$\leq O(n^{-r/2}) \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} n^i O(n^{j/2}) = O\left(n^{(r/2)-s}\right)$$

$$= o(1) \quad (\text{for } s > r/2)$$

Now, since $\varepsilon > 0$ is arbitrary, it follows that $I_3 = o(1)$. Also, $I_4 \to 0$ as $n \to \infty$ and hence $I_2 = o(1)$, combining the estimates of I_1 and I_2, we obtain (3.1).

To prove the uniformity assertion, it sufficient to remark that $\delta(\varepsilon)$ in above proof can be chosen to be independent of $x \in [a,b]$ and also that the other estimates holds uniformly in $[a,b]$.

■

Our next theorem is a Voronovaskaja-type asymptotic formula for the operators $S_{n,\nu}^{(r)}(f;x),\ r = 1, 2, \ldots$.

Theorem 3.2. Let $f \in C_\alpha[0,\infty)$ for some $\alpha > 0$. If $f^{(r+2)}$ exists at a point $x \in (0,\infty)$,

then

$$\lim_{n\to 0} n\left(S_{n,\nu}^{(r)}(f;x) - f^{(r)}(x)\right) = r(r+1)f^{(r)}(x) + \{(r+1-\nu) + 2x(r+1)\} f^{(r+1)}(x) + x(1+x) f^{r+2}(x). \qquad (3.3)$$

Further, if $f^{(r+2)}$ exists and is continuous on the interval $(a-\eta, b+\eta) \subset (0,\infty),\ \eta > 0$, then (3.3) holds uniformly on $[a,b]$.

Proof: By the Taylor's expansion of $f(t)$, we get

$$S_{n,\nu}^{(r)}(f;x)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}S_{n,\nu}^{(r)}\left((t-x)^i;x\right)+S_{n,\nu}^{(r)}\left(\varepsilon(t,x)(t-x)^{r+2};x\right)$$

$$:=I_1+I_2,$$

where $\varepsilon(t,x)\to 0$ as $t\to x$.

By Lemma 2.2 and (3.2), we get

$$I_1=\sum_{i=r}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}S_{n,\nu}^{(r)}(t^j;x)$$

$$=\frac{f^{(r)}(x)}{r!}S_{n,\nu}^{(r)}(t^r;x)+\frac{f^{(r+1)}(x)}{(r+1)!}\left((r+1)(-x)S_{n,\nu}^{(r)}(t^r;x)+S_{n,\nu}^{(r)}(t^{r+1};x)\right)$$
$$+\frac{f^{(r+2)}(x)}{(r+2)!}\left(\frac{(r+2)(r+1)}{2}x^2S_{n,\nu}^{(r)}(t^r;x)+(r+2)(-x)S_{n,\nu}^{(r)}(t^{r+1};x)+S_{n,\nu}^{(r)}(t^{r+2};x)\right)$$

$$=f^{(r)}(x)\left(\frac{(n+r-1)!(n-r-2)!}{(n-1)!(n-2)!}\right)+\frac{f^{(r+1)}(x)}{(r+1)!}\left\{(r+1)(-x)\left(\frac{(n+r-1)!(n-r-2)!}{(n-1)!(n-2)!}r!\right)\right.$$
$$\left.+\left(\frac{(n+r)!(n-r-3)!}{(n-1)!(n-2)!}(r+1)!x+\frac{(n+r-1)!(n-r-3)!}{(n-1)!(n-2)!}(r+1)(r+1-\nu)r!\right)\right\}$$
$$+\frac{f^{(r+2)}(x)}{(r+2)!}\left\{\frac{(r+1)(r+2)}{2}x^2\left(\frac{(n+r-1)!(n-r-2)!}{(n-1)!(n-2)!}r!\right)\right.$$
$$+(r+2)(-x)\left(\left(\frac{(n+r)!(n-r-3)!}{(n-1)!(n-2)!}\right)(r+1)!x+\left(\frac{(n+r-1)!(n-r-3)!}{(n-1)!(n-2)!}\right)(r+1)(r+1-\nu)r!\right)$$
$$\left.+\left(\frac{(n+r+1)!(n-r-4)!}{(n-1)!(n-2)!}.\frac{(r+2)!}{2}x^2+\frac{(n+r)!(n-r-4)!}{(n-1)!(n-2)!}(r+2)(r+2-\nu)!(r+1)!x\right)\right\}+o(1)$$

Hence in order to prove (3.3) it suffices to show that $nI_2\to 0$ as $n\to\infty$, which follows on proceeding along the lines of proof of $I_2\to 0$ as $n\to\infty$ in Theorem 3.1.

The uniformity assertion follows as in the proof of Theorem 3.1. ■

Finally, we present a theorem which gives as an estimate of the degree of approximation by $S_{n,\nu}^{(r)}(f;x)$ for smooth functions.

Theorem 3.3. Let $f\in C_\alpha[0,\infty)$ for some $\alpha>0$ and $r\le q\le r+2$. If $f^{(q)}$ exists and is continuous on $(a-\eta,b+\eta)\subset(0,\infty),\eta>0$, then for sufficiently large n,

$$\left\|S_{n,\nu}^{(r)}(f;x)-f^{(r)}(x)\right\|_{C[a,b]}\le C_1n^{-1}\sum_{i=r}^{q}\left\|f^{(i)}\right\|_{C[a,b]}+C_2\,n^{-1/2}\omega_{f^{(q)}}\left(n^{-1/2}\right)+O(n^{-2})$$

where C_1, C_2 are constants independent of f and n, $\omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$, and $\|.\|_{C[a,b]}$ denotes the sup-norm on $[a,b]$.

Proof. By a finite Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)+h(t,x)(1-\chi(t)),$$

where ξ lies between t,x and $\chi(t)$ is the characteristic function of the interval $(a-\eta,b+\eta)$ and

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i$$

Now,

$$S_{n,\nu}^{(r)}(f;x)-f^{(r)}(x)=\left(\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_{n,\nu}^{(r)}(t,x)(t-x)^i\,dt-f^{(r)}(x)\right)$$

$$+\int_0^\infty W_{n,\nu}^{(r)}(t,x)\left\{\frac{f^{(q)}(\xi)-f^{(q)}(x)}{q!}(t-x)^q\chi(t)\right\}dt$$

$$+\int_0^\infty W_{n,\nu}^{(r)}(t,x)h(t,x)(1-\chi(t))\,dt$$

$$:=I_1+I_2+I_3.$$

By using Lemma 2.2 and (3.2), we get

$$I_1=\sum_{i=r}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=r}^{i}\binom{i}{j}(-x)^{i-j}\frac{d^r}{dx^r}\left(\frac{(n+j-1)!\,(n-j-2)!}{(n-1)!(n-2)!}x^j\right.$$

$$\left.+\frac{(n+j-2)!\,(n-j-2)!}{(n-1)!(n-2)!}j(j-\nu)x^{j-1}+O(n^{-2})\right)-f^{(r)}(x).$$

Consequently,

$$\|I_1\|_{C[a,b]}\le C_1n^{-1}\left(\sum_{i=r}^{q}\left\|f^{(i)}\right\|_{C[a,b]}\right)+O(n^{-2}),\text{ uniformly on }[a,b].$$

To estimate I_2 we proceed as follows:

$$|I_2|\le\int_0^\infty\left|W_{n,\nu}^{(r)}(t,x)\right|\left\{\frac{\left|f^{(q)}(\xi)-f^{(q)}(x)\right|}{q!}|t-x|^q\chi(t)\right\}dt$$

$$\le\frac{\omega_{f^{(q)}}(\delta)}{q!}\int_0^\infty\left|W_{n,\nu}^{(r)}(t,x)\right|\left(1+\frac{|t-x|}{\delta}\right)|t-x|^q\,dt$$

$$\le\frac{\omega_{f^{(q)}}(\delta)}{q!}\left[(n-1)\sum_{k=\nu}^{\infty}\left|p_{n,k}^{(r)}(x)\right|\int_0^\infty p_{n,k-\nu}(t)\left(|t-x|^q+\delta^{-1}|t-x|^{q+1}\right)dt\right.$$

$$+\sum_{k=0}^{r-1} p_{n,k}^{(r)}(x)\left(|x|^q+\delta^{-1}|x|^{q+1}\right)\Bigg], \delta>0.$$

Now, for $s=0,1,2,\ldots$, using Schwartz inequality for integration , summation, (2.1) and Lemma 2.3 we have

$$(n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)|k-nx|^j \int_0^\infty p_{n,k-\nu}(t)|t-x|^s d \le (n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)|k-nx|^j \left\{\left(\int_0^\infty p_{n,k-\nu}(t)dt\right)^{1/2}\right.$$

$$\left.\times\left(\int_0^\infty p_{n,k-\nu}(t)(t-x)^{2s}dt\right)^{1/2}\right\}$$

$$\le \left(\sum_{k=\nu}^{\infty} p_{n,k}(x)(k-nx)^{2j}\right)^{1/2}\left((n-1)\sum_{k=\nu}^{\infty} p_{n,k}(x)\int_0^\infty p_{n,k-\nu}(t)(t-x)^{2s}\right)^{1/2}$$

$$=O(n^{j/2})O(n^{-s/2})$$

$$=O(n^{(j-s)/2}), \text{ uniformly on } [a,b]. \qquad (3.4)$$

Therefore, by Lemma 2.4 and (3.4), we get

$$(n-1)\sum_{k=\nu}^{\infty}\left|p_{n,k}^{(r)}(x)\right|\left|\int_0^\infty p_{n,k-\nu}(t)|t-x|^s dt\right| \le (n-1)\sum_{k=\nu}^{\infty}\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i|k-nx|^j\frac{|Q_{i,j,r}(x)|}{x^r}p_{n,k}(x)$$

$$\times\int_0^\infty p_{n,k-\nu}(t)\,|t-x|^s dt$$

$$\le\left(\sup_{\substack{2i+j\le r\\ i,j\ge 0}}\sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r}\right)\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\left(\sum_{k=\nu}^{\infty}p_{n,k}(x)|k-nx|^j\int_0^\infty p_{n,k-\nu}(t)|t-x|^s dt\right)$$

$$=C\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i O(n^{(j-s)/2})=O(n^{(r-s)/2}), \text{uniformly on } [a,b]. \qquad (3\cdot 5)$$

$$\left(\text{since } \sup_{\substack{2i+j\le r\\ i,j\ge 0}}\sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^r}:=M(x) \text{ but fixed}\right)$$

Choosing $\delta=n^{-1/2}$ and applying (3.5), we are led to

$$\|I_2\|_{C[a,b]}\le\frac{\omega_{f^{(q)}}(n^{-1/2})}{q!}\left[O(n^{(r-q)/2})+n^{1/2}O(n^{(r-q-1)/2})+O(n^{-m})\right], \text{(for any } m>0)$$

$$\le C_2\, n^{-(r-q)/2}\omega_{f^{(q)}}(n^{-1/2}).$$

Since $t\in[0,\infty)\setminus(a-\eta,b+\eta)$, we can choose $\delta>0$ in such a way that $|t-x|\geq\delta$ for all $x\in[a,b]$. Thus, by Lemmas 2.3 and 2.4 , we obtain

$$|I_3|\leq(n-1)\sum_{k=\nu}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}n^i|k-nx|^j\frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r}p_{n,k}(x)\int_{|t-x|\geq\delta}p_{n,k-\nu}(t)|h(t,x)|\,dt$$

$$+\sum_{k=0}^{\nu-1}\sum_{\substack{2i+2j\leq r\\ i,j\geq 0}}n^i|k-nx|^j\frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r}p_{n,k}(x)|h(0,x)|$$

For $|t-x|\geq\delta$, we can find a constant C such that $|h(t,x)|\leq C|t-x|^{\alpha}$. Hence, using Schwarz inequality for integration and then for summation ,(2.1)and (2.3), it easily follows that $I_3=O(n^{-s})$ for any $s>0$, uniformly on $[a,b]$.

Combining the estimates of $I_1, I_2\, I_3$, the required result is immediate. ■

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تعميم مؤثر باسكوكوف فيلبس للمجموع التكاملي

علي جاسم محمد
جامعة البصرة - كلية التربية -قسم الرياضيات

امل خليل حسن
جامعة البصرة - كلية العلوم - قسم الرياضيات

الخلاصة

في هذا البحث ، نقدم وندرس صيغة معممة من مؤثرات باسكاكوف فيلبس للمجموع التكاملي.
نبرهن ان هذه المؤثرات متقاربة للدالة المقربة . كذلك نناقش صيغة فروفنسكي المشابهة ونوجد الخطأ المخمن بواسطة معيار الاستمرارية لهذه المؤثرات.