# Rational Solution of the General Modified Camassa-Holm Degasperis -Procesi Equation

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## الخلاصة

في هذا البحث جرت دراسة معادلة ( gm-CH-DP ) والتي هي تعميم للمعادلتين MCH و MDP حيث تم استخدام حركة الاقطاب في المستوي العقدي لايجاد حلاً نسبياً لها . وقد تم اشتقاق النتائج لمعادلتي MCH و MDP من خلال عملنا هذا .

## ABSTRACT

In this paper, we studied the gm-CH-DP equation by using the dynamics of the poles of it's rational solution in the complex x- plane, in order to find the rational solution of gm-CH-DP equation. In this work the results of the MCH , MDP equations are derived through our work.

## INTRODUCTION

The Camassa-Holm equation (CH) of the form

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad \dots (1)$$

was derived as a shallow water wave with surface tension in an asymptotic expansion that extends one order beyond the Korteweg - deVries (KdV) equation [8] while the Degasperis – Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \quad \dots (2)$$

was originally derived by Degasperis – Procesi using the method of asymptotic integrability . Both equations (1) and (2) are contained in the family of equations, [ 6 ]

$$u_t - u_{xxt} + (a+1)uu_x = au_x u_{xx} + uu_{xxx} \quad \dots (3)$$

Mikhailov and Novikov [ 10 ] developed a powerful extension of the symmetry classification method, and applied this to the equation (3) . They found that only the cases a=2,3 could possess infinitely many commuting symmetries , and so only these two cases are integrable . In a recent paper Degasperis et. al constructed a Lax pair of the equation (2) , and hence proved the integrability of the Degasperis – Procesi equation
[ 5] .

Since the CH and DP equation have rich structures , Wazwaz [2] suggested a modified form of the Camassa – Holm equation (called MCH) ,

$$u_t - u_{xxt} + 3u^2 u_x = 2u_x u_{xx} + uu_{xxx} \quad \dots (4)$$

and a modified form of the Degasperis–Procesi equation (called MDP),

$$u_t - u_{xxt} + 4u^2 u_x = 3u_x u u_{xx} + u u_{xxx} \qquad \ldots (5)$$

In [11] , Liang and Jeffrey was obtained two rational types of solutions to Modified Camassa-Holm equation which are :

$$(i)\ u_1(x,t) = \frac{8}{x^2}$$

$$(ii)\ u_2(x,t) = \frac{8}{(x-3t)^2} - 1$$

And for Modified Degasperis-Procesi equation were :

$$(i)\ u_1(x,t) = \frac{15}{2x^2}$$

$$(ii)\ u_2(x,t) = \frac{15}{2(x-4t)^2} - 1$$

In this paper , we investigated a general form of the MCH and MDP and study the motion of the poles of their rational solutions . The rest of this paper is splitted into three sections , in section 2 we illustrate the soliton solution of the gm-CH-DP equation while in section 3 we investigate the rational solution and the motion of the poles of the equation . Finally . in section 4 , the conclusions of this work is presented.

## The soliton solution of the (gm-CH-DP) equation :

The general modified Camassa – Holm- Degasperis – Procesi (gm-CH-DP) equation is of the form [ 1] :

$$q_t - q_{xxt} + (a+1)q^2 q_x = a q_x q_{xx} + q q_{xxx} \qquad \ldots (6)$$

where a is any real number . This equation is used in the study of shallow water dynamics and is integrable [ 2 ] . The solitary wave solution of this equation first appeared in 2006 . The 1- soliton solution of (6) is given by [ 2]

$$q(x,t) = \frac{A}{\cosh^2 B(x - \bar{x})}$$

where

$$A = \frac{-3(a+2)}{2(a+1)} \quad , \quad B = \frac{1}{2} \qquad \ldots (7)$$

Here A represents the amplitude of the solitons while B is the inverse width of the soliton and $\bar{x}$ represents the center position of the soliton and therefore the velocity of the soliton is given by

$$v=\frac{d\bar{x}}{dt} \qquad \dots (8)$$

Using the meromorphic expansion [4 ] :

$$\operatorname{cosech}^2 x=\sum_{n=-\infty}^{\infty}\frac{1}{(x+in\pi)^2} \qquad \dots (9)$$

( uniformly convergent except at the point $x=in\pi$ ) one easily obtains the following meromorphic expansion for the one – soliton solution of the general modified Degasperis – procesi Camassa – Holm equation :

$$q(x,t)=A\sum_{n=-\infty}^{\infty}\frac{1}{(B(x-\bar{x})+\frac{i\pi}{2}+in\pi)^2} \qquad \dots (10)$$

## The motion of the poles of rational solutions :

In this section , we look for a rational solution for gm-CH-DP equation . By the expression (10) , we take a rational solution of the form[1] :

$$q(x,t)=\sum_{k=1}^{N}\frac{R_k(t)}{(x-x_k(t))^2} \qquad N\in Z^+ \qquad \dots(11)$$

where N is the number of poles , we concentrate on the pole $x_1$ $x=x_1+\in$ ,with $\in>0$ we have different cases :

Case (i) : If N = 1 , then we have :

$$q(x,t)=\frac{R_1(t)}{(x-x_1)^2}=\frac{R_1}{\in^2} \qquad R_1\neq 0 \qquad \dots(12)$$

$$q_x=\frac{-2R_1}{\in^3} \qquad \dots(13)$$

$$q_{xx}=\frac{6R_1}{\in^4} \qquad \dots(14)$$

$$q_{xxx}=\frac{-24R_1}{\in^5} \qquad \ldots(15)$$

$$q_t=\frac{\dot{R_1}}{\in^2}+\frac{2R_1\dot{x_1}}{\in^3} \qquad \ldots(16)$$

$$q_{xxt}=\frac{6\dot{R_1}}{\in^4}+\frac{24R_1\dot{x_1}}{\in^5} \qquad \ldots(17)$$

Substitute the equations (12-17) in equation (6) , we have :

$$\frac{\dot{R_1}}{\in^2}+\frac{2R_1\dot{x_1}}{\in^3}-\frac{6\dot{R_1}}{\in^4}-\frac{24R_1\dot{x_1}}{\in^5}-2(a+1)\frac{R_1^3}{\in^7}=-12a\frac{R_1^2}{\in^7}-24\frac{R_1^2}{\in^7}$$

Then equating coefficients of power of $\in$ to zero , we have :

$$R_1=\frac{6(a+2)}{(a+1)} \quad \text{and} \quad \dot{x}_1=0 \qquad \ldots(18)$$

Hence , we exclude the case a = -2,-1 .

Case (ii) : If N > 1, then we can rewrite q in (11) as

$$q(x,t)=\frac{R_1(t)}{\in^2}+F(\in,t) \qquad \ldots(19)$$

$$q_x=\frac{-2R_1(t)}{\in^3}+\frac{\partial F}{\partial\in} \qquad \ldots(20)$$

$$q_{xx}=\frac{6R_1(t)}{\in^4}+\frac{\partial^2 F}{\partial\in^2} \qquad \ldots(21)$$

$$q_{xxx}=\frac{-24R_1(t)}{\in^5}+\frac{\partial^3 F}{\partial\in^3} \qquad \ldots(22)$$

$$q_t=\frac{\dot{R}_1(t)}{\in^2}+\frac{2R_1(t)\,\dot{x}_1(t)}{\in^3}+G(\in,t) \qquad \ldots(23)$$

$$q_{xxt}=\frac{6\dot{R}_1(t)}{\in^4}+\frac{24R_1(t)\,\dot{x}_1(t)}{\in^5}+\frac{\partial^2 G}{\partial\in^2} \qquad \ldots(24)$$

where

$$F(\in,t)=\sum_{k=2}^{N}\frac{R_k(t)}{(x_1+\in-x_k(t))^2} \qquad \ldots (25)$$

$$G(\in,t)=\sum_{k=2}^{N}\frac{\dot{R}_k(t)}{(x_1+\in-x_k(t))^2}+\frac{2R_k(t)\,\dot{x}_k(t)}{(x_1+\in-x_k(t))^3} \qquad (26)$$

For sake of simplicity , we shall expand the functions F , G and their derivatives with respect to $\in$ , then substitute (19-24) in (6) , we have :

$$\frac{\dot{R}_1(t)}{\in^2}+\frac{2R_1(t)\,\dot{x}_1(t)}{\in^3}+G(0,t)+\in G^{/}(0,t)+\frac{\partial^2 G(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)$$

$$-\frac{6\,\dot{R}_1(t)}{\in^4}-\frac{24R_1(t)\,\dot{x}_1(t)}{\in^5}-G^{//}(0,t)-\in G^{///}(0,t)-\frac{\partial^2 G^{//}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}$$

$$+O(\in^3)+(a+1)\left\{\frac{-2R_1^3(t)}{\in^7}-\frac{4R_1^2(t)}{\in^5}\left[F(0,t)+\in F^{/}(0,t)+\frac{\partial^2 F(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\right]\right.$$

$$-\frac{2R_1(t)}{\in^3}\left[F^2(0,t)+2\in F(0,t)F^{/}(0,t)+\frac{\partial^2 F^2(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\right]+\frac{R_1^2(t)}{\in^4}$$

$$\left[F^{/}(0,t)+\in F^{//}(0,t)+\frac{\partial^2 F^{/}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\right]+\frac{2R_1(t)}{\in^2}\Big[F(0,t)+$$

$$\in F^{/}(0,t)+\frac{\partial^2 F(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\Big]\Big[F^{/}(0,t)+\in F^{//}(0,t)+\frac{\partial^2 F^{/}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\Big]+$$

$$\left.\Big[F^2(0,t)+2\in F(0,t)F^{/}(0,t)+\frac{\partial^2 F^2(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\Big]\Big[F^{/}(0,t)+\in F^{//}(0,t)+\frac{\partial^2 F^{/}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\Big]-\right\}$$

$$+\frac{12a\,R_1^2(t)}{\in^7}+\frac{2a\,R_1(t)}{\in^3}\Big[F^{//}(0,t)+\in F^{///}(0,t)+\frac{\partial^2 F^{//}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\Big]+$$

$$\frac{24R_1^2(t)}{\in^7}+\frac{R_1(t)}{\in^2}\Big[F^{////}(0,t)+\in F^{////}(0,t)+\frac{\partial^2 F^{////}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\Big]-$$

$$\frac{6a\,R_1(t)}{\in^4}\Big[\ F^{/}(0,t)+\in F^{//}(0,t)+\frac{\partial^2 F^{/}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\ \Big]-a\Big[\ F^{/}(0,t)+\in F^{//}(0,t)+\frac{\partial^2 F^{/}(\in,t)}{\partial\in^2}\Big|_{\in=0}+O(\in^3)\Big]$$

$$+\frac{24R_1(t)}{\in^5}\Big[\ F(0,t)+\in F^{/}(0,t)+\frac{\partial^2 F(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\ \Big]+\Big[\ F(0,t)+\in F^{/}(0,t)+\frac{\partial^2 F(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\ \Big]$$

$$\frac{\partial^2 F^{/}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\ \Big]\Big[\ F^{//}(0,t)+\in F^{///}(0,t)+\frac{\partial^2 F^{//}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\ \Big]$$

$$\Big[\ F^{///}(0,t)+\in F^{////}(0,t)+\frac{\partial^2 F^{///}(\in,t)}{\partial\in^2}\Big|_{\in=0}\frac{\in^2}{2!}+O(\in^3)\ \Big]=0 \qquad \ldots(27)$$

Then equating coefficients of $\in^n$ to zero give some information about the unknowns :

The coefficient for $\in^{-7}$ leads to :

$$R_1(t)=\frac{6(a+2)}{(a+1)} \qquad a\neq -1 \qquad \ldots(28)$$

While the coefficient for $\in^{-5}$ gives the dynamics of the first pole :

$$\dot{x}_1(t)=-(a+1)\,F(0,t) \qquad \ldots(29)$$

and for $\in^{-3}$, we have :

$$F^{//}(0,t)=\frac{-(a+1)}{5a}F(0,t)\,(\,1+F\ (0,t)) \qquad \ldots(30)$$

Coefficient of $\in^{-4}$ leads to $F^{/}(0,t)=0 \quad or \quad a=\frac{-1}{2}$

Let $a\neq\frac{-1}{2}$ and substitute $F^{/}_{\in}(0,t)=0$ in the remaining coefficients gives that $F^{//}_{\in}=F^{///}_{\in}=F^{////}_{\in}=F^{v}_{\in}=0$

So , by this way we get the solutions :

$$q(x,t)=\frac{R_1(t)}{\in^2}+F(0,t) \qquad \ldots(31)$$

Which can be written as :

$$q_1(x,t)=\frac{6(a+2)}{(a+1)}*\frac{1}{(x-x_1(t))^2} \qquad \ldots(32)$$

and

$$q_2(x,t) = \frac{6(a+2)}{(a+1)} * \frac{1}{(x-x_1(t))^2} - 1 \qquad \ldots(33)$$

And the dynamical system for the motion of the poles are become :

$$\dot{x}_1(t) = 0 \qquad \text{Or} \qquad \dot{x}_1(t) = (a+1) \qquad \ldots(34)$$

In this paper, we study the rational solutions and the motion of the poles of general Modified Camassa Holm Degasperis – Procesi equation, then we noted the following three important points :

**First :** the rational solution of modified Camassa Holm and Degasperis – procesi equation [17] are special case of rational solution of gm-CH-DP equation such that :

**a.** If $a=2$ , $x_1(t)=0$ , $F(0,t)=0$ , we obtain the first type of rational solution of MCH .

**b.** If $a=2$ , $x_1(t)=3t$ , $F(0,t)=-1$ , we obtain the second type of rational solution of MCH .

**c.** If $a=3$ , $x_1(t)=0$ , $F(0,t)=0$ , we obtain the first type of rational solution of MDP .

**d.** If $a=3$ , $x_1(t)=4t$ , $F(0,t)=-1$ , we obtain the second type of rational solution of MDP .

**Second :** we can indicate that the value of a depend on the coefficient of the term ( $u_x u_{xx}$) but the value of the second pole ( $x_2(t)$) depend on the coefficient of ( $u^2 u_x$) .

**Third :** the value of $F(0,t)$ depend on the value of poles such that if the pole equal to zero then $F(0,t)$ have the value zero too and if the pole equal to any value except zero , the function $F(0,t)$ have the vaule (-1).

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