Approximate Symmetries for Nonlinear Diffuion Equation

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الخلاصة

أن الهدف من العمل هو دراسة وإيجاد التماثلات التقريبية لمعادلة الانتشار غير الخطية $h_t = (\frac{1}{3}h^3 h_r)_r$ عند تشويشها أولا باستخدام الحد $(\frac{\in}{r}h^3 h_r)$ ثم باستخدام داله مجهولة $\frac{\in}{r}f(h,h_r)$ لقد قمنا بإيجاد حل تقريبي لامتغاير في الحالة الأولى ووصف اللاخطية $f(h,h_r)$ في الحالة الثانية.

ABSTRACT

The aim of this work is to study and obtain approximate symmetries to the nonlinear diffusion equation,

$$h_t = (\frac{1}{3}h^3 h_r)_r$$

when it is perturbed , first by the term $(\frac{\in}{r}h^3 h_r)$ and second by the unknown function $\frac{\in}{r}f(h,h_r)$. An approximate invariant solution is found in the first case, while we descried all nonlinearities $f(h,h_r)$ in the second case.

INTRODUCTION

The concept of approximate symmetry goes back to at least the late seventies when the observation was made that, since the differential equations which arise in mathematical modeling are invariably approximate, one should in fact be considering appro-ximate symmetry [1]

The non-linear diffusion equation having the form

$$h_t = (D(h)h_r^n)_r \qquad (1)$$

for a single function h of two independent variables t and r; $D(h)$ is the diffusion term, has a wide range of applications in physics, diffusion process and engineering sciences [2],[3].

In this paper, we study approximate symmetry for the nonlin-ear diffusion equation (1) , when n=1 $D(h) = \frac{1}{3}h^3$ and the equation is perturbed. When the perturbed term is $\frac{\in}{r}h^3 h_r$ we find an approximate invariant solution while when the perturbed term is $\frac{\in}{r} f(h,h_r)$, where f is an unknown function, we describe all nonlinearities $f(h,h_r)$ so that the equation is approximately invariant, where $\in$ is a small parameter. Such a type of equation arises in important geophysics in the spreading of molten lava on the surface of the earth and in the drop Medicine [4]

The plan of this paper is as follows:
in section two, we introduce the approximate symmetry. In section three, perturbation by the term $\frac{\in}{r} h^3 h_r$ is studied. While in section four, we study perturbation by an unknown function $f(h,h_r)$

Approximate Symmetry Approach

To study a differential equation containing a small param-eter $\in$, symmetries are found in a form of series in powers of $\in$ as

$$X = X_0 + \in X_1 + \in^2 X_2 \dots$$

The first summand is just a symmetry of the unperturbed equation (i.e. for an equation with $\in = 0$).Taking a few summands means finding approximate symmetry.

Let us begin by recalling X_0, the symmetries of the unpertu-rbed equation

$$h_t = h^2 h_r^2 + \frac{1}{3} h^3 h_{rr} \qquad (2)$$

which has been found (see[5]), as

$$X_0 = \xi_1^0(r,t,h)\ \partial r + \xi_2^0(r,t,h)\partial t + \eta^0(r,t,h)\partial h$$

where

$$\xi_1^0 = \left(\frac{1}{2}\alpha_1 + \alpha_2\right) r + \alpha_3,\ \ \xi_2^0 = (\alpha_1 t + \alpha_4),\ \eta^0 \frac{2}{3} h\alpha_2$$

α_i, i = 1, 2, 3, 4 are arbitrary constants

First Perturbation

When equation (2) is perturbed by the term $\frac{\in}{r} h^3 h_r$, it becomes:

$$h_t = h^2 h_r^2 + \frac{1}{3} h^3 h_{rr} + \frac{\in}{r} h^3 h_r \qquad (3)$$

the first-order approximate symmetry is

$$X = X_0 + \in X_1$$

where

$$X_1 = \xi_1^1(r,t,h)\partial r + \xi_2^1(r,t,h)\partial t + \eta^1(r,t,h)\partial h \qquad (4)$$

The criterion for invariance of the PDE (3) is

$$X^{[2]}\left(h_t - h^2h_r^2 - \frac{1}{3}h^3h_{rr} - \frac{\in}{r}h^3h_r\right)\Bigg|_{h_t h^2 h_r^2 + \frac{1}{3}h^3 h_{rr} + \frac{\in}{r}h^3 h_r} = 0$$

which implies that,

$$\frac{1}{r}\left(\frac{2}{3}\alpha_2 - \alpha_1\right)h^2h_r + \varsigma_1^t - 2hh_r^2\eta^1 - 2h^2h_r\varsigma_1^r - h^2h_{rr}\eta^1$$
$$-\frac{1}{3}h^3\varsigma_1^{rr} + \frac{1}{r^2}h^3h_r\left(\left(\frac{1}{2}\alpha_1 + \alpha_2\right)r + \alpha_3\right) - \frac{3}{r}h^3h_r\left(\frac{2}{3}h\alpha_2\right) \qquad (5)$$
$$-\frac{1}{r}h^3\left(\frac{2}{3}\alpha_2 - \frac{1}{2}\alpha_1 - \alpha_2\right)h^3 = 0$$

when

$$h_t = h^2h_r^2 + \frac{1}{3}h^3h_{rr} + \frac{\in}{r}h^3h_r$$

We obtain a polynomial equation in h_{rr}, h_{rt}, h_r, which must hold for arbitrary values of r, t, h, h_r, h_{rr}, h_{rt}. From the coefficients of h_{rt}, h_r h_t, h_r $h_{rr,}$ we get

$$\xi_1^1 = (\frac{1}{2}\delta_1 + \delta_2)r + \delta_3 \quad \xi_2^1 = \delta_1 t + \delta_4 \quad \eta^1 = \frac{2}{3}h\delta_2$$

where δ_1, …. , δ_4 are constants

Invariant Solution

In order to obtain a group-invariant solution of (3) we use the Lagrange Method of Lie symmetry generators as follows:

Let

$$\gamma_1 = \left(\frac{1}{2}\alpha_1 + \alpha_2\right) + \in\left(\frac{1}{2}\delta_1 + \delta_2\right)$$

$$\gamma_2 = \alpha_1 + \in\delta_1$$

$$\gamma_3 = \alpha_4 + \in\delta_4 \qquad , \gamma_4 = \frac{2}{3}\left(\alpha_2 + \in\delta_2\right)$$

then

$$\frac{dr}{\gamma_1 r + \in\delta_3} = \frac{dt}{\gamma_2 t + \in\gamma_3} = \frac{dh}{\gamma_4 h} \qquad (6)$$

whose solution is given by

$$s = (r + \beta_1)(t + \beta_3)^{-\beta_2}, \qquad h = (t + \beta_3)^{\beta_4} f(s) \qquad (7)$$

where $\beta_1 = \frac{\in\delta_3}{\gamma_1}$, $\beta_2 = \frac{\gamma_1}{\gamma_2}$, and $\beta_3 = \frac{\in\delta_3}{\gamma_2}$.

Differentiate (7) with respect to t and r, then substitute in equation (3) we get

$$\beta_4(t+\beta_3)^{\beta_4-1} - f(s) - \beta_2(t+\beta_3)^{\beta_4-\beta_2-1}(r+\beta_1)f'(s) - (t+\beta_3)^{4\beta_4-2\beta_2}$$
$$f^2(s)f'^2(s) - \frac{1}{3}(t+\beta_3)^{4\beta_4-2\beta_2} f^3(s)f''(s) - \frac{\in}{r}(t+\beta_3)^{4\beta_4-\beta_2} f^3(s)f' = 0,$$

and to be able to proceed further we make the assumption that $\beta_1 = 0$ find simple equation and multiply by $r(t+\beta_3)^{\beta_2-4\beta_4}$, we get

$$\beta_4 sf(s) - \beta_2 s^2 f'(s) - sf^2(s)f'^2(s) - \frac{s}{3}f^3(s)f''(s) - \in f^3(s)f'(s) = 0$$

if $\beta_4=+2$, $\beta_2=-1$,then

$$\frac{d}{ds}\left(s^2 f(s)\right) + \left[\frac{1}{3}s\frac{d}{ds}\left(f^3(s)f'(s)\right) + \in f^3(s)f'(s)\right] = 0$$

integrating with respect to s gives

$$s^2 f(s) + \left[\frac{1}{3}\left(sf^3(s)f'(s) - \int f^3(s)f'(s)ds\right) + \in \int f^3(s)f'(s)ds\right] = C_1$$

where C_1 is arbitrary constant. If $C_1 = \frac{1}{3} - \in = 0$, we get

$$f = \left(-\frac{9}{2}s^2 + C_2\right)$$

consequently the approximate invariant solution is

$$h(r,t) = (t+\beta_3)^2\left(\frac{-9}{2}s^2 + C_2\right)^{\frac{1}{3}} \qquad (8)$$

Second Perturbation

In this section we consider equation (2) when it is perturbed by an unknown function $\frac{\in}{r}f(h,h_r)$,which becomes,

$$h_t = \frac{1}{3}\frac{\partial}{\partial r}\left(h^3 h_r\right) + \frac{\in}{r}f(h,h_r) \qquad (9)$$

We would like to find the transformation that will have the property to change any element of the family of PDE's (9) to PDE which belongs to the same family.

In order to obtain the group of transformation of equation (9), the following auxiliary equations are imposed [5]

$$f_t = f_r = f_{ht} = 0 \qquad (10)$$

and the approximate generator is given in the form of

$$X = \left(\xi_1^0 + \in \xi_1^1\right)\partial r + \left(\xi_2^0 + \in \xi_2^1\right)\partial t + \left(\eta^0 + \in \eta^1\right)\partial h + \psi\partial f$$

where $\left(\xi^v, T^v, \eta^v\right)(v = 0,1)$ are functions of t, r and h; and f is a function of h, h_r only and ψ function of t, r, h and h_r.

The second prolongation of the operator X is

$$X^{[2]} = X + \left(\varsigma_0^r + \in \varsigma_1^r\right)\partial h_r + \left(\varsigma_0^t + \in \varsigma_1^t\right)\partial h_t + \left(\varsigma_0^{rr} + \in \varsigma_1^{rr}\right)\partial h_{rr} .$$
$$\psi_h \partial f_h + \psi_{h_r} \partial f_{h_r}$$

The infinitesimal invariance criterion for equation (10) becomes

$$X^{[2]}\left(h_t - h^2 h_r^2 - \frac{1}{3} h^3 h_{rr} - \frac{\in}{r} f(h,h_r) \right)\Bigg|_{h_t h^2 h_r^2 + \frac{1}{3}h^3 h_{rr} + \frac{\in}{r} f(h,h_r)} = 0$$

this equation yields

$$\left(\varsigma_0^t + \in \varsigma_1^t\right) - 2hh_r^2\left(\eta^0 + \in \eta^1\right) - 2h^2 h_r\left(\varsigma_0^r + \in \varsigma_1^r\right) - h^2 h_{rr}$$
$$\left(\eta^0 + \in \eta^1\right) - \frac{1}{3}h^3\left(\varsigma_0^{rr} + \in \varsigma_1^{rr}\right) + \frac{\in}{r^2}\xi^0 f - \frac{\in}{r}\psi = 0 \qquad (11)$$

In zero-order approximation ($\in$=0), (11) yields the invariant criterion in the form of

$$\varsigma_0^t - 2hh_r^2\eta^0 - 2h^2 h_r \varsigma_0^r - h^2 h_{rr}\eta^0 - \frac{1}{3}h^3 \varsigma_0^{rr} = 0 \quad , \in \psi = 0 \qquad (12)$$

which gives $\psi = 0$ $\forall \theta \in (\{r, t, h, h_r\})$, since f is a differential variable which is algebraically independent of f_h and f_{hr}. Thus ψ is the function of h, h_r and f only.

It is clear that (12) is the same determining equation as in the case of an unperturbed equation (2),and so its infinitesimal coefficients are

$$\xi_1^0 = \left(\frac{1}{2}\alpha_1 + \alpha_2\right) r + \alpha_3, \quad \xi_2^0 = \alpha_1 t + \alpha_4, \quad \eta^0 = \frac{2}{3} h \alpha_2$$

where α_i, i = 1, 2, 3, 4 are constants defined by the following generators

$$X_1^0 = \partial r \ , \ X_2^0 = \partial t \ , \ X_3^0 = \frac{1}{2} r \partial r + t \partial t \ , \ X_4^0 = r \partial r + \frac{2}{3} h \partial h$$

In a similar way to the above, we have the invariance condition

$$\varsigma_1^t - 2hh_r^2\eta^1 - 2h^2 h_r \varsigma_1^r - h^2 h_{rr}\eta^1 - \frac{1}{3}h^3 \varsigma_1^{rr} + \frac{1}{r^2}\xi_1^0 f - \frac{1}{r}\psi = 0$$

when

$$h_t = h^2 h_r^2 + \frac{1}{3}h^3 h_{rr} + \frac{\in}{r} f(h, h_r)$$

Thus, we obtain a polynomial equation in h_{rr}, h_{rt}, h_r, which must hold for arbitrary values of r, t, h, h_r, h_{rr}, h_{rt}.

From the coefficients we get the following results

$\alpha_3 = 0$

$\xi_2^1 = \gamma_1 t + \gamma_4$

$\xi_1^1 = \gamma_2 r + \gamma_3$

$\eta^1 = \dfrac{(2\gamma_2 - \gamma_1)h}{3}$

and

$$\psi=\left(\frac{4}{3}\alpha_2-\frac{1}{2}\alpha_1\right)f$$

where γ_i, i = 1, 2, 3, 4, are constants. Therefore the generator X will be

$$X=\left(\left(\frac{1}{2}\alpha_1+\alpha_2\right)r+\in(\gamma_2 r+\gamma_3)\right)\partial r+((\alpha_1 t+\alpha_3)+\in(\gamma_1 t+\gamma_4))\partial t$$

$$+\frac{2}{3}(\alpha_2+\gamma_4)h\partial h$$

we observe that there are seven approximate symmetries.

$$X_1^0=\partial t \qquad X_2^0=\frac{2}{3}h\partial h+r\partial r+\frac{4}{3}f\partial f \qquad X_3^0=\frac{1}{2}r\partial r+t\partial t-\frac{1}{2}f\partial f$$

$$X_1^1=\in\partial r \qquad X_2^1=\in\partial t \qquad X_3^1=\in\left(\frac{2}{3}h\partial h+r\partial r\right)$$

$$X_4^1=\in\left(t\partial t-\frac{1}{3}h\partial h\right)$$

The Set of Transformations

Consider now the approximate transformation corresponding to the generators X_i^j, i=1,2,3,4 ;j=1,2

These transformations are given by the "exponentiation" formula [6] as follows

$$r^*=e^{\in X}r=\sum_{k=0}^{\infty}\frac{\in^k}{k}X^k r$$

we get the results (up to $\in^1$) then

$$r^*=r+\in Xr$$

$$r^*=r+\in\left(\left(\frac{1}{2}\alpha_1+\alpha_2\right)+\in\gamma_2\right)r=\left(1+\in\left(\frac{1}{2}\alpha_1+\alpha_2\right)+\in^2\gamma_2\right)r$$

in a same way we find

$$t^*=t+\in(\alpha_1+\in\gamma_1)t=(1+\in(\alpha_1+\in\gamma_1))t$$

$$h^*=h+\in\left(\frac{2}{3}\left(\alpha_2+\in\left(\frac{2\gamma_2-\gamma_1}{3}\right)\right)\right)h=\left(1+\frac{2}{3}\in\left(\frac{2\gamma_2-\gamma_1}{3}\right)\right)h$$

$$f^*=f+\in\left(\frac{4}{3}\alpha_2\frac{1}{2}\alpha\right)f=\left(1+\in\left(\frac{4}{3}\alpha_2-\frac{1}{2}\alpha_1\right)\right)f.$$

Conclusion

This paper deals with finding the symmetries of equation $h_t=(\frac{1}{3}h^3h_r)_r+\in F(r,h,h_r)$ using the invariance of the equation when it is unperturbed ($\in=0$) instead of finding the symmetries directly .The first perturbation is made by a known function $\frac{\in}{r}h^3h_r$ .concerning the second perturbation in which there is a class of DEs, one has to choose those changes of variables which do not alter the form of the class of PDE(9).

This study may be continued to study the convergence of the approximate symmetry either by finding the exact symmetry to the perturbed equation (one can compare with [4] when to $\in=\frac{1}{3}$) or use numerical methods, so that can obtain the error value

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