Connection Between Painlevé Property and Hirota's Bilinearization

Inaam A. Malloki Imad A. Barzinjy
Department of Mathematics, University of Al-Mustansirya, Baghdad, Iraq
Department of Mathematics, University of Arbil, Arbil, Iraq
Received 13/1/2009 – Accepted 16/3/2009

الخلاصة

النتيجة وتعديل بينلفيه وخاصية هيروتا طريقة بين العلاقة دراسة هو البحث هذا من الأساسي الهدف ان لمعادلة 1985 عام في وجماعته جيبون قبل من الموضوعة KdV من المشتق باكلاند تحويل ان بينا لقد عدة على تطبق ان يمكن المعدلة الطريقة وان هيروتا طريقة من نظامية بطريقة ايجاده يمكن بينلفية خاصية معادلة منها معادلات B, KP, CDG . نتائجنا برهنة في مساعدة خواص باشتقاق ايضا قمنا البحث هذا في الرئيسية

ABSTRACT

The main goal of this paper is to consider the connection between Hirota's method and Painlevé property based on a result obtained by Gibbon et al. in 1985 to KdV equation. Our contribution is a modification to their work, so that Bäcklund transformation derived from Painlevé property can be obtained in a systematic way from Hirota's bilinear form.

We find that the modified procedure can be applied to many equations. Among these are Boussinesq (B), KP and CDG equations.

Also in this paper, some useful properties for the bilinear operators are derived to prove our main results.

INTRODUCTION

There is no rigid definition determined yet considering (integrability) of nonlinear partial differential equations, but the following properties are well accepted:

1- The PDE possess the Painlevé property.[1,2,3,4,5]

2- Existence of bilinear form.[3,4]

3- Solvability by IST.[2,3,4]

4- The ability to linearize the nonlinear PDEs by an explicit transformation.[1,5]

5- Existence of N-soliton solution, i.e, the equation possesses solitary waves, an infinite number of conservation laws.[1,3,4,5]

6- The equation satisfies the Ablowitz–Ramani–Segur (ARS) conjecture.[2,5]

In particular, we concentrate on Painleve_ property and Hirota's bilinear form.

In 1983 Weiss proposed a new test of the complete integrability of nonlinear partial differential equations which was based on the generalization of the Painleve_ property known formerly only for ordinary differential equations (ODEs) in 1941. The connection between ODEs of the painleve__ type and the completely integrability PDEs has been pointed out by Ablowitz and Segur in 1977 [6].

Let us discuss the Painleve_ property for some special nonlinear partial differential equations like KdV, KP, B and CDG equations.

A partial differential equation has Painleve_ property when its solution is single-valued about the movable, singularity manifold. To be precise, if the singularity manifold is determined by

$$\varphi(z_1, z_2, \ldots, z_n) = 0 \quad (1.1)$$

and $u = u(z_1, z_2, \ldots, z_n)$ is a solution of the PDE, then the assumption

$$u = \varphi^{\alpha} \sum_{j=0}^{\infty} u_j \varphi^j, \quad u_0 \neq 0 \quad (1.2)$$

where

$$\varphi = \varphi(z_1, z_2, \ldots, z_n), \quad u_j = u_j(z_1, z_2, \ldots, z_n), \quad j = 0,1,2,\ldots$$

are analytic functions of (z_j) in a neighborhood of the manifold (1.1) and α is an integer. The derivation of Painleve property is illustrated through the known KdV equations,

$$u_t + 12 u u_x + u_{xxx} = 0, \quad (1.\ 3)$$

To find the Painleve property for this equation we substitute (1.2) in (1.3) then we find that,

$$\alpha = -2$$

and

$$u = \varphi^{-2} \sum_{j=0}^{\infty} u_j \varphi^j \quad (1.4)$$

The recursion relations for the u_j's are presented in [7]. It is found that the "resonances" occur at $j = -1, 4, 6$.

The compatibility conditions at j = 4,6 are satisfied identically and the KdV equation possesses the Painleve_ property. For instance, when

$$j = 0: \quad u_0 = -\varphi_x^2,$$

1 , 1: xx j = u = φ

2 : 12 4 3 2 0 ,

2

= + 2 + − = x t x x xxx xx j φ φ φ u φ φ φ

3: 12 0 , 3

2

2 = + − + = xt xx x xxxx j φ φ u φ u φ (1.5)

(12) 0

4 : :

3

2

2 + + − =

∂

∂

=

u u

x

j compatibility condition

xt xxxx xx x φ φ φ φ

(1.6)

By (1.5), the compatibility condition (1.6) at j = 4 is satisfied identically.

The compatibility condition at j = 6 involves extensive calculation and is also satisfied identically [8]. We now specialize (1.4) by setting the "resonance" functions

0 . 4 6 u = u =

Furthermore, by requiring

0 , 3 u =

u_2 must satisfy (1.3) and consequently it is easily demonstrated that

u = 0 , j ≥ 3 , j

Thus we obtain the following Bäcklund transforms

2 2 ,

2

ln u

x

u +

∂

∂

= φ (1.7)

and

(,) 12 4 3 2 0

2

2

1 2 = + + − = x t x x xxx xx B u φ φ φ φ u φ φ φ (1.8)

(,) 12 0 2 2 2 = + + = xt xx xxxx B u φ φ φ u φ . (1.9)

For Boussinesq (B) equation,

0 .

3

+ 2 + 2 2 + 1 = tt xx x xxxx u uu u u (1.10)

The associated Bäcklund transformations are

2 2 ,

2

2 ln u

x

u +

∂

∂

= φ (1.11)

and

(,) 2 2 0

3 2

2 2 4

1 2 = − + + = t xx x xxx x B u φ φ φ φ φ u φ (1.12)

(,) 2 0 3 2

1

2 2 B u = + + u = tt xxxx xx φ φ φ φ . (1.13)

Bäcklund transformation for KP equation,

+ 6 2 + 6 + + = 0 tx x xx xxxx yy u u uu u u (1.14)

are

2 2 ,

2

2 ln u

x

u +

∂

∂

= φ (1.15)

and

(,) 4 3 6 2 0 ,

2

2 2

1 2 = + − + + = t x x xxx xx y x B u φ φ φ φ φ φ φ u φ (1.16)

(,) 6 0 . 2 2 2 B u = + + + u = $_{xt\ xxxx\ yy\ xx}$ φ φ φ φ φ (1.17)

while for the CDG equation,

(+ 30 + 60 $_3$)= 0

∂

∂

+ u uu u

x

u $_{t\ xxxx\ xx}$ (1.18)

are

2 2 ,

2

ln u

x

u +

∂

∂

= φ (1.19)

and

(( )) 30 6 4 3 0

(,) 6 15 10

2

2

2

2

2

2,

2

2

1 2

+ + + − =

= + − + +

$x\ xx\ x\ x\ xxx\ xx$

$t\ x\ xxxxx\ x\ xx\ xxxx\ xxx$

u u u

B u

φ φ φ φ φ

φ φ φ φ φ φ φ φ

(1.20)

(,) 30(6) 0 2

2

2 2 2, 2 B u = + + u + u + u = $_{xt\ xxxxxx\ xx\ xx\ xx\ xxxx}$ φ φ φ φ φ φ (1.21)

where u and u_2 satisfy (1.18)

An important contribution, which we shall discuss next, has been made by Hirota who invented a novel means of calculating N-soliton solutions with its attendant implication of integrability without having to resort to other method [3]. It is very powerful method of simplifying algebraic calculations [9].

The Hirota's bilinear formalism has been instrumental in derivation of the multisoliton solution of (integrable) nonlinear evolution equations

[10]. Using the bilinear operator D_s which acts on a pair of functions (the so-called "dot product) antisymmetrically as:

$$D_s(f\cdot g) = \left(\frac{\partial}{\partial s} - \frac{\partial}{\partial s'}\right) f(s)\, g(s')\Big|_{s'=s}$$

where s denotes x,t or y.

The main steps of Hirota's method are:

First: The selection of a suitable substitution instead of the dependent variable that allows obtaining the bilinear form of the equation.

Second: The consideration of the formal series of perturbation theory for this bilinear equation. In the case of soliton solutions these series are terminated [11].

In this paper, the procedure as described in the first step of Hirota's method is applied to the equations investigated previously. If we use the transformation

2 2 2 2 ln (,) f $f x t f f f$

x

u_{xxx} −

=

∂

∂

= (1.22)

then Hirota's bilinear form for KdV eq.(1.3) is

$$\left(D_x^4 + D_x D_t\right)(f.f) = 0 \quad (1.23)$$

and for B eq.(1.10) is

$$\left(3D_t^2 + D_x^4\right)(f.f) = 0 \quad (1.24)$$

for KP eq.(1.14) is

$$\left(D_x^4 + D_x D_t + D_y^2\right)(f.f) = 0 \quad (1.25)$$

and finally for CDG eq. (1.18) is

$$\left(D_x D_t + D_x^6\right)(f.f) = 0. \quad (1.26)$$

An equation $K[u]=0$ is defined as the KdV–type equation by Peterson [12] if there exist two nontrivial Hirota's polynomials $P=P(\mu,\nu.\omega)$ and $N=N(\mu,\nu.\omega)$ such that

$$P(D_x)\theta\cdot\theta = 0 \Rightarrow K[\theta^{-2} N(D_x)\theta\cdot\theta] = 0$$

holds for all nontrivial positive function $\theta = \theta(x)$.

It can be easily seen that our equations (KdV, Bousinesq, KP and CDG) are of KdV–type according to Peterson's definition.

Note that if $N(D_x)(f \cdot f) =$ then we say that f satisfies Hirota's bilinear form.

For the KdV equation, Hirota's method and Painlevé property are connected [3] in the following way: We can think of u and u_2, in the Bäcklund transformation (1.7), as an adjacent pair of solution in a set $\{u_n\}$ which are related to Hirota's f-function f_n by

$$u_n = (\ln f_n)_{xx} \quad (1.27)$$

Relabelling the u, u_2 and φ in (1.7) as u_n, u_{n-1} and φ_{n-1} respectively, then the following theorem can be proved [3]

Theorem: If the functions f_n (n =1,2,3,…) satisfy the Hirota's equation (1.23), for every n, and if

$$f_{n-1} = \varphi_{n-1} f_n \quad (1.28)$$

then the resulting equation in φ_{n-1}

φ and u_{n-1} is satisfied by the painlevé result (1. 8) and (1.9). Furthermore

∏-

=

=

1

0

n

i

$_{n\,i} f$ φ

Gibbon et al. also claimed that similar theorem for NLS and mKdV equation can be proved.

Note that Gibbon's et al. connection is restricted to KdV equation.

In this paper we consider some equations in the class of KdV–type equations (in the sense of Peterson) in which the dependent variable transformation used is

$(\;)_{xx}\, u = \ln f$ (1.29)

to obtain some Hirota's polynomials,

$P(D\, ,D\, ,D\, ,...)(f\,.\,f) = 0$ $_{x\,t\,y}$ (1.30)

Our modification is to obtain Bäcklund transformation for each equation.

USEFUL PROPERTIES OF HIROTA'S OPERATOR

For each equation which will be investigated in this paper the following properties are satisfied.

Note:

In the following derivations we put f instead of $f(x,t)$, f' instead of $f(x',t')$ and so as to φ and φ ′ .

For the functions φ and f, It can be easily proved that

1. $_2(φ \cdot φ\,) = φ\; _{2\,2}(\;\cdot\;) + \;_{2\,2}(φ \cdot φ\,)\; _{x\,x\,x} D\, f\, f\, D\, f\, f\, f\, D$.

2. $_4(φ \cdot φ\,) = φ\; _{2\,4}(\;\cdot\;) + 6\; _2(\;\cdot\;)\; _2(φ \cdot φ\,) + \;_{2\,4}(φ \cdot φ\,)\; _{x\,x\,x\,x\,x} D\, f\, f\, D\, f\, f\, D\, f\, f\, D\, f\, D$

3.

$_6(φ \cdot φ\,) = φ_2\; _6(\;\cdot\;) + 15\; _4(\;\cdot\;)\; _2(φ \cdot φ) + 15\; _2(\;\cdot\;)\; _4(φ \cdot φ) + \;_{2\,6}(φ \cdot φ)\; _{x\,x\,x\,x\,x\,x\,x} D\, f\, f\, D$

$f\, f\, D\, f\, f\, D\; D\, f\, f\, D\, f\, D$

4. $(φ \cdot φ\,) = φ\; _2(\;\cdot\;) + \;_2(φ \cdot φ\,)\; _{x\,t\,x\,t\,x\,t} D\; D\, f\, f\, D\; D\, f\, f\, f\, D\; D$.

5. (a) If $_{xx}\, u = a(\ln f)$ then $2\; u\, f\; _2 D_2(f\, f)$

$a\; _x \cdot\; = \cdot$ where a is non zero constant.

(b) If $_{xx}\, u = (\ln f)$ then $2(u\; u_2)\, f\; _2 D_4(f\, f)\; _{x\,x} + \; = \; \cdot$

T HE MAIN RESULTS

In this section, we present the connection between Painlevé property and Hirota's bilinear form for KdV, KP, B and CDG equations.

Theorem 1:

If each function f, f_2 satisfies the Hirota's bilinear form for KdV equation (1. 3) and if there exists a function φ such that

$$f_2 = \varphi f \quad (3.1)$$

then

$$B_1(u, \varphi^2) - B_2(u, \varphi^2) = 0$$

where $B_1(u, \varphi) = 0$ and $B_2(u, \varphi) = 0$ are the Bäcklund transformation (1.8) and (1.9) respectively.

Proof: Since f satisfies Hirota's bilinear form for KdV then we have

$$\left(D_x^4 + D_x D_t\right)(f \cdot f) = 0 \quad (3.2)$$

substituting the transformation (3.1) in (3.2) gives,

$$\left(D_x^4 + D_x D_t\right)(\varphi^2 f \cdot \varphi^2 f) = 0$$

and by linearity,

$$D_x^4(\varphi^2 f \cdot \varphi^2 f) + D_x D_t(\varphi^2 f \cdot \varphi^2 f) = 0$$

Now using properties (2 and 4) we get

( ) ( ) ( ) ( )
( ) ( )0
6
2
2 2 2
2
2 4
2
2
2 2
2
2 2
2 4
+ · + · =
· + · · + · +
φ φ φ
φ φ φ φ φ
x t x t
x x x x
D D f f f D D

$D f f D f f D f D$

or,

$(\ ) (\ ) (\ ) (\ ) (\ )$

$(\ )$

$2\,4\,2\,4\,2\,2$

$2\,2\,2\,2\,2\,2\,2$

2

2

6

0

$x\,x\,t\,x\,x\,x$

$x\,t$

$D f f D D f f f D \ D f f D$

$f D \ D$

$\varphi \ \varphi \ \varphi \ \varphi \ \varphi$

$\varphi \ \varphi$

$[[\cdot + \cdot]] + \cdot + \cdot \cdot$

$+ \cdot =$

Since f_2 satisfies Hirota's bilinear form for KdV then we have

$(\ )(\ ) \ 0 \ 2\,2$

$D_4 + D \ D f \cdot f = {}_{x\,x\,t}$

which implies (by property (5(a)) that $2 \ (\ ) \ 2\,2$

$2\,2$

${}_{2\,2} u f D f f_x = \cdot$

then

$(\ ) \ 12 (\ ) \ {}_2 (\ ) \ 0$

2

$2\,2$

$2\,2$

$2\,4$

${}_2 \varphi \cdot \varphi + \varphi \cdot \varphi + \varphi \cdot \varphi = {}_{x\,x\,x\,t} f D \ u f D f D D$

or,

$(\ ) \ 12 \ {}_2 (\ ) \ (\ ) \ 0$

2

${}_4 \varphi \cdot \varphi + \varphi \cdot \varphi + \varphi \cdot \varphi = {}_{x\,x\,x\,t} D \ u \ D \ D \ D$

Now using properties (1), (2) and (4) we get

$2 (\ 4 \ 3 \ ) \ 12 \ [2 (\ {}_2)] \ 2 (\ ) \ 0$

2

2

${}_4 - + + - + - = {}_{x\,x\,xxx\,xx\,xx\,x\,xt\,x\,t} \varphi\varphi \ \varphi \ \varphi \ \varphi \ u \ \varphi\varphi \ \varphi \ \varphi\varphi \ \varphi \ \varphi$

This can be written as

$(\ 12 \ ) \ (\ 12 \ {}_2 4 \ 3 \ {}_2) \ 0$

${}_{2\,2} + + - + + - = {}_{xt\,xx\,xxxx\,x\,t\,x\,x\,xxx\,xx} \varphi \ \varphi \ u \ \varphi \ \varphi \ \varphi \ \varphi \ u \ \varphi \ \varphi \ \varphi \ \varphi$

This identity can be considered as a polynomial in φ ,which gives, the

Bäcklund transformation,

(,) 12 2 4 3 2 0

1 2 2 = + + − = $x t x x xxx xx$ B u φ φ φ u φ φ φ φ

and,

(,) 12 0 2 2 2 = + + = $xt\ xx\ xxxx$ B u φ φ u φ φ

for KdV equation.

Theorem 2:

If each function f, f_2 satisfies the Hirota's bilinear form for Boussinesq equation (1.10) and if there exists a function φ such that

$_2 f$ = φ f (3.3)

then

(,) (,) 0 2 2 1 2 φ B u φ − B u φ =

where (,) 0 1 2 B u φ = and (,) 0 2 2 B u φ = are the Bäcklund transformation (1.12) and (1.13) respectively.

Theorem 3:

If each function f, f_2 satisfies the Hirota's bilinear form for KP equation (1.14) and if there exists a function φ such that

$_2 f$ = φ f (3.4)

then

(,) (,) 0 2 2 1 2 φ B u φ − B u φ =

where (,) 0 1 2 B u φ = and (,) 0 2 2 B u φ = are the Bäcklund transformation (1.16) and (1.17) respectively

Theorem 4:

If each function f, f_2 satisfies the Hirota's bilinear form for CDG equation (1.18) and if there exists a function φ such that

$_2 f$ = φ f (3.5)

then

(,) (,) 0 2 2 1 2 φ B u φ − B u φ =

where (,) 0 1 2 B u φ = and (,) 0 2 2 B u φ = are the Bäcklund transformation (1.20) and (1.21) respectively.

The Proof of each theorem 2, 3 or 4 is similar to the proof of theorem 1.

CONCLUDING REMARKS

In this work, we established a connection between the two concepts of integrability of nonlinear PDEs, the Painlevé property and

Hirota's bilinear form through Gibbon's et al. result which needs to be modified in order:

a. To find algorithmically their associated Bäcklund transformation. In Gibbon's et al theorem for KdV equation there is infinite number of functions satisfying Hirota's bilinear form. In fact, there is no need to use all of them. It is sufficient to use the last u (in the expansion) whose value is not arbitrary for the resulting equation of Painlevé property.

b. To consider Hirota's bilinearization as an alternative technique to Painlevé property to obtain the Bäcklund transformation.

c. To apply Gibbon's et al. result to many equations, not only KdV equation. For this purpose, we adopt Peterson's definition and the procedure as described for KdV has been modified to be applied to many equations like B, KP and CDG equations.

REFERENCES

1.Costakis, S. and Leach, P. G. L., Symmetries, Singularities and Integrability in Nonlinear Mathematical Physics and Cosmology, Proceedings of Institute of Math. of NAS of Ukraine, 2002, V. 43, Part 1, 128–135.

2.Estevez, P. G., Conde, E. and Gordoa, P. R., Unified Approach to Miura, Bäcklund and Darboux Transformations for Nonlinear Partial Differential Equations, J. Math. Phys., Vol. 5, N1, 1998, 82-114.

3.Carstea, A. S., Extension of the Bilinear Formalism to Supersymmetric KdV-Type Equation, solv-int/9812022 v1, 1998.

4.Kondo, K., Studies on Integrability for Nonlinear Dynamical Systems and its Applications, Osaka Univ., Toyonaka, Osaka 560–8531, Japan, 2001.

5.Muestte, M., Painleve_ Analysis for Nonlinear Partial Differential Equations, The Painleve_ property , one century later, ed. R. Conte, CRM series in mathematical physics, Springer-Verlag, Berlin, 1998.

6.Doktorov, E. V. and Sakovich, S. Yu., Painleve_ Test and Integrability of Nonlinear Klein-Fock-Gordon Equation, J. Phys. A: Math. Gen. 18, 1985, 3327-3334

7. Weiss, J., The Painleve_ Property for Partial Differential Equations. II BäcklundTransformation, Lax Pairs, and the Schwarzian Derivative, J. Math. Phys. 24, 1983, 1405-1413.

8.Weiss, J., The Painleve_ Property for Partial Differential Equations,La Jolla Institute Report LJI-R-82-186, 1982.

9.Ramani, A., Inverse Scattering, Ordinary Differential Equation of Painleve_ Type and Hirota's Bilinear Formalism, Thesis, Univ. of Paris–Sud, 1981.
10.Gibbon, J. D. and Tabor M., Hirota's Method and Painlevé Property, Imperial College, London and Columbia University, New York, 1985
11.Hietarinta, J., Introduction to Hirota Bilinear Method, Lecture Notes in Physics, Springer Berlin/ Heideiberg vol 495/ 1997.
12. Peterson P.,Multi-Soliton Interactions and the Inverse Problem of Wave Cresets, Ph.D Thesis Tallin Technical University, Estona, 2001.