Poles' Order of Rational Solutions for Evolution Equations

Inaam A. Malloki
Department of mathematics, College of Science University of Al-Mustanairyah

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الخلاصة

بمساعدة مبرهنة نشر الكسر الجزيئي فان هذا البحث هو محاولة لايجاد طريقة لتوحيد الفرضيات الحلول النسبية للمعادلات التفاضلية الجزئية باسلوب اعتيادي بسيط. لقد برهنت عدة نتائج جديدة لمعادلات خطية وغير خطية. وكذلك اعطينا خوارزمية كفوءة لايجاد رتبة الاقطاب للحل. واخيرا فان تطبيقات قد وضعت. طريقة لمعادلات خطية ذات الرتبة الاولى ولمعادلة m-KdV

ABSTRACT

With help of partial fraction expansions theorem ,this paper is an attempt to provide a unifying scheme to the assumptions of rational solutions of partial differential equations, in a natural and simple way. Some new results for linear and nonlinear PDEs have been proved.
Moreover, we give an efficient algorithm for obtaining the poles' order for the solution. Finally, applications of the approach for 1st order linear PDE and the m-KdV equation are presented.

INTRODUCTION

There are many tools which can be used to evaluate domain of analyticity of solutions to nonlinear partial differential equations (PDEs)[1]. Among them is the method of pole dynamics originated with Kruskal's work [2]. F. Calogero [3] in 1977 has discussed the motion of the poles of special solutions of the celebrated KdV equation. He has investigated the possibility that this nonlinear evolution differential equation admits (possibly complex) solutions which are, for all time, rational in x. Choodnovskys [4] displayed the connection between some PDEs and many – particle systems introducing pole expansions of certain solutions and showing that the time evolution of the position of poles corresponds to the motion of classical particles. Senouf in [1] showed how to use pole dynamics to determine the evolution of the domain of regularity of the solution of Burgers' equation with a generic initial data.

Through the rational solution to the PDE the pole dynamics has been discussed by many authors: In 1983, Kametaka [5] discussed the rational similarity solutions, using the zeros of Yablonski or Vorobiev polynomials. Consideration of the Bäcklund transformations of modified equations, Weiss has proved a method for iterative construction of rational solutions [6].

Dealing with the examination of nonlinear PDEs with solutions which have movable singularities of first order, Kudryashov has demonstrated multiphase and rational solutions of the family of nonintegrable equations [7].Liz and Schwarz had looked for rational solutions of Riccati nonlinear PDEs [8]. Rational solutions of the viscous Burgers' equation and for KdV equation are examined by Deconinck et. al[9,10], using the dynamics of their poles in the complex x-plane. Rational solutions of the KdV and m-KdV equations and rational and rational-oscillatory solutions of the nonlinear Schrödinger equation are expressed in terms of special polynomials by Clarkson in [11], and for the classical Boussinesq equation in [12].

This paper deals with evolution equations of power nonlinearities which have rational solutions with finite number of poles. We investigate the problem of finding the poles' order for some classes of PDEs. In particular, linear PDEs and the generalized Fishers and KdV equations. An efficient algorithm to compute the order is presented. The work is based mainly on the following theorem of partial fraction of expansion of a meromorphic function with simple poles :

Theorem

Let f be a meromorphic function with only simple poles $z_1, z_2...$ with $|z_1| \le |z_2| \le ...$ Let $\text{Res}_{z_n} f(z) = b_n$ for each *n*. Let ζ_k be a sequence of nested squares with vertices $a_k(\pm 1, \pm i)$ such that $a_k \to \infty$ as $k \to \infty$ which do not pass through any pole of *f*. Suppose that f is analytic at 0 and that there exists $M \in R^+$, independent of *k* such that $|f(z)| \le M$ on ζ_k for each *k*. Then except at the poles of *f*.

$$f(z) = f(0) + \sum_{n=1} b_n \left(\frac{1}{z - z_n} + \frac{1}{z_n}\right)$$

This result can be generalized to include non-simple poles.

Proof: see [13].

The paper has been organized as follows.Section 2 is entirely devoted to showing how to find the order of the poles.In section 3, we present an algorithm to compute this order, while section 4 is devoted to applications to this approach. The final section contains the discussion and concluding remarks.

THE MAIN RESULTS

Let us consider the evolution equations with power nonlinearities,

$$\frac{\partial u}{\partial t} = V[u] \qquad (1)$$

Here we consider real scalar functions u of two variables x, t $\in$ R and differential operators V of the form $V[u] \equiv V(x, u, \frac{\partial u}{\partial x}, ..., \frac{\partial^p u}{\partial x^p})$. Let us investigate the possibility that (1) admits (possibly complex) solutions which are, for all time, rational in *x*. To unify many previous results of many authors on construction of rational solution, and with aid of partial fraction expansions theorems, we assume that a solution u (x, t) of (1) is meromorphic with finite number (N) of poles of the same order (n) in the x-plane. Suppose that u is analytic at (0, t), $t > 0$ and the pole x_k and it's resedue R_k are functions of t. Thus u takes the form,

$$u(x,t) = u(0,t) + \sum_{k=1}^{N} R_k(t)\left(\frac{1}{(x - x_k(t))^n} + \frac{1}{x_k(t)}\right)$$

or

$$u(x,t) = \sum_{k=1}^{N} \frac{R_k(t)}{(x - x_k(t))^n} + \lambda(t) \qquad (2)$$

where

$$\lambda(t) = u(0,t) + \sum_{k=1}^{N} \frac{R_k(t)}{x_k(t)}$$

The aim of this study is to find the natural number n (the order of all poles). When the form (2) is substituted in the PDE (1), the derivatives will be :

$$u_t = \sum_{k=1}^{N} \frac{\dot{R}_k}{(x - x_k)^n} + n\sum_{k=1}^{N} \frac{R_k \dot{x}_k}{(x - x_k)^{n+1}} + \dot{\lambda}(t) \qquad , \quad \dot{} = \frac{d}{dt}$$

and

$$u_{x^r} = A_{n,r} \sum_{k=1}^{N} \frac{R_k}{(x - x_k)^{n+r}}$$

where $A_{n,r} = (-1)^r n(n+1)\ldots(n+r-1) \quad , \; r = 1,2\ldots,p$

Note: the analysis will be carried out for the first pole since it is the same for all poles.

When x belongs to a neighbor of x_1, i.e $x = x_1 + \varepsilon$, $\varepsilon \neq 0$ each derivative in the PDE is considered as a function of ε as follows:

$$u = \frac{R_1}{\varepsilon^n} + F(\varepsilon) \qquad (3)$$

$$u_t = \frac{\dot{R}_1}{\varepsilon^n} + \frac{nR_1\dot{x}_1}{\varepsilon^{n+1}} + G(\varepsilon) \qquad (4)$$

and

$$u_{x^r} = A_{n,r}\frac{R_1}{\varepsilon^{n+r}} + H_r(\varepsilon) \quad , \quad r = 1,2,...,p \tag{5}$$

where the functions F, G and H_r are the analytic functions of ε:

$$F(\varepsilon) = \sum_{k=2}^{N}\frac{R_k}{(x_1+\varepsilon-x_k)^n} + \lambda(t) \tag{6}$$

$$G(\varepsilon) = \sum_{k=2}^{N}\left(\frac{\dot{R}_k}{(x_1+\varepsilon-x_k)^n} + \frac{nR_k\dot{x}_k}{(x_1+\varepsilon-x_k)^{n+1}}\right) + \dot{\lambda}(t)$$

(7)

$$H_r(\varepsilon) = A_{n,r}\sum_{k=2}^{N}\frac{R_k}{(x_1+\varepsilon-x_k)^{n+r}} \quad , \quad r = 1,2,...,p \tag{8}$$

Lemma (1)
Equation (1) has no rational solution if V is linear and $p \geq 2$.
Proof: V can be written as

$$V = \frac{\partial^p u}{\partial x^p} + \alpha_1\frac{\partial^{p-1}u}{\partial x^{p-1}} + ... + \alpha_{p-1}\frac{\partial u}{\partial x} + \alpha_p u + \alpha_{p+1}, \quad p \geq 2$$

where $\alpha_1,...,\alpha_p,\alpha_{p+1}$ are constants.
Substituting the derivatives formulas 3, 4 and 5 in the equation (1), we have,

$$\frac{\dot{R}_1}{\varepsilon^n} + \frac{nR_1\dot{x}_1}{\varepsilon^{n+1}} + G = A_{n,p}\frac{R_1}{\varepsilon^{n+p}} + H_p + \alpha_1(A_{n,p-1}\frac{R_1}{\varepsilon^{n+p-1}} + H_{p-1}) + ... + \alpha_p(\frac{R_1}{\varepsilon^n} + F) + \alpha_{p+1}$$

It is clear that the highest nonpositive power of ε is $n+p$ whatever n. Equating to zero the coefficient of $\varepsilon^{-(n+p)}$ leads to the result that $R_1 = 0$, which means that equation (1) has no rational solution.□

Two classes of PDEs will be investigated now.
The first class is the generalized Fisher's equation [7],

$$u_t = \alpha_1 u + \alpha_2 u^2 + \alpha_3 u^3 + \beta u_x + u_{xx} + \gamma u u_x \tag{9}$$

where $\alpha_1,\alpha_2,\alpha_3,\beta,$ and γ are constants
Rational solutions of equation (9) can be classified in the following theorem,
Theorem 1.

(1) The poles of rational solution of equation (9) are simple, if $\alpha_3 \neq 0$ or $\gamma \neq 0$.
(2) The poles of rational solution of equation (9) are double, if $\alpha_3 = \gamma = 0$ and $\alpha_2 \neq 0$.
(3) Equation (9) have no rational solution, if $\alpha_2 = \alpha_3 = \gamma = 0$.

Proof :

Substituting 3, 4 and 5 in equation (9), we have

$$\frac{\dot{R}_1}{\varepsilon^n}+\frac{nR_1\dot{x}_1}{\varepsilon^{n+1}}+G=\alpha_1(\frac{R_1}{\varepsilon^n}+F)+\alpha_2(\frac{R_1}{\varepsilon^n}+F)^2+\alpha_3(\frac{R_1}{\varepsilon^n}+F)^3$$

$$+\beta(A_{n,1}\frac{R_1}{\varepsilon^{n+1}}+H_1)+(A_{n,2}\frac{R_1}{\varepsilon^{n+2}}+H_2)+\gamma(\frac{R_1}{\varepsilon^n}+F)(A_{n,1}\frac{R_1}{\varepsilon^{n+1}}+H_1)$$

The nonpositive power of ε play a crucial role in obtaining the value of n. It is clear that the highest nonpositive powers of ε are, $N_1=n+2$ and possibly $N_2=3n$ or $N_3=2n+1$ (depending on α_3 or γ vanishing). If $\alpha_3=\gamma=0$ then we have unique maximum power which leads to $R_1=0$, while, if $\alpha_3\neq 0$ or $\gamma\neq 0$ then we have at least two maximum candidates. Consequently, and for the seek of rational solution, the first step is to equate at least two maximum candidates $N_1=N_2(N_3)$ which leads to $n=1$, i.e (1) holds. To prove (2), if $\alpha_3=\gamma=0$ then take the second maximum candidates $N_1=n+2$ and $N_2=2n$ if $\alpha_2\neq 0$. $N_1=N_2$ implies that $n=2$. Now to prove (3), if $\alpha_2=\alpha_3=\gamma=0$ then equation (9) becomes 2nd order linear PDE which is according to lemma 1, has no rational solution.□

The second class to be considered, is the generalized KdV equation [7]

$$u_t=\alpha_1 u+\alpha_2 u^2+\alpha_3 u^3+\alpha_4 u^4+\beta_1 u_{xx}+u_{xxx}+\gamma_1 uu_x+\gamma_2 uu_{xx}+\gamma_3 u^2 u_x+\gamma_4 u_x^2 \quad (10)$$

where $\beta_1,\alpha_i,\gamma_i, i=1,2,3,4$ are constants.

Theorem 2 classifies rational solutions for equation (10) :

Theorem 2.

1. If at least one of the coefficients $\alpha_4,\gamma_2,\gamma_3$ or γ_4 is not equal to zero then the poles of rational solution of equation (10) are simple.
2. If $\alpha_4=\gamma_2=\gamma_3=\gamma_4=0$ and $\gamma_1\neq 0$ then the poles of rational solution of equation (10) are double.
3. If $\alpha_4=\gamma_1=\gamma_2=\gamma_3=\gamma_4=0$ and $\alpha_3\neq 0$ then equation(10) has no rational solution.
4. If $\alpha_3=\alpha_4=\gamma_1=\gamma_2=\gamma_3=\gamma_4=0$ and $\alpha_2\neq 0$ then equation(10) has rational solution with poles of order 3.
5. If $\alpha_2=\alpha_3=\alpha_4=\gamma_1=\gamma_2=\gamma_3=\gamma_4=0$ then equation(10) has no rational solution.

Proof: As in theorem 1, the proof consists of several stages depending on the coefficients in equation (10). Begin with equating the 1st maximum possible candidates gives (1). If the conditions in (1) are not satisfied, take the 2nd maximum possible candidates to get (2), and similarly for (3) and (4). Finally the equation becomes linear which ends the proof.□

The following algorithm helps in calculating n :

THE ALGORITHM

We present here the method for utilizing the assumption (2) in the PDE as an algorithm. This algorithm tells if there is a possibility of rational solutions to the PDE or not and provides us with the order of their poles if they exist. The whole analysis of determining the poles' order is the nonpositive powers of $\varepsilon = x - x_1$. Hence, instead of considering the whole equation, the algorithm deals with numbers derived from the PDEs' terms. There are three steps in this algorithm:

Setp 1 : Assign to each derivative u_{s^r} (in the PDE) the number ($n + r$), where $s = x$ or t; $r = 0,1,2,...,p$.

Step 2 : Assign to each term (in the PDE) a natural number obtained by adding the numbers (from step 1) of the derivatives in that term. As an example, assign $2n + (n+1) + (n+2)$ to the term $u^2 u_x u_{xx}$.

Step 3 : Choose $N_1,...,N_k$ to be the candidates for the maximum number of the natural numbers (from step 2) for the PDE's terms.

(i) If $k = 1$, then there is no rational solution to the PDE,

(ii) If $k \neq 1$, and $N_1 \equiv N_2 \equiv ... \equiv N_k$, then set n=1,and

(iii) If $k \neq 1$, then the order n is the solution of the equations $N_i = N_j \quad i, j = 1,2,...,k$, if it is a unique natural number. Otherwise there is no rational solution.

APPLICATIONS

1. First order linear PDE

Let us consider the problem

$$u_t = u_x + \beta u + \gamma, \quad u(0,t) = \phi(t)$$

The general solution of the PDE can be obtained using Lagranges' auxiliary system. We use our algorithm to find the solution of the problem.

Step 1,2: The natural numbers assigned to each term are as follows:

$$\underset{n \quad n+1}{u_t} = \underset{n+1}{\underset{\downarrow}{u_x}} + \beta \underset{n}{\underset{\downarrow}{u}} + \gamma$$

Step 3: It is clear that $N_1 = n+1, \quad N_2 = n+1$ are the maximum candidates. But $N_1 \equiv N_2, \quad$ hence we set n=1.

Consequently

$$u(x,t) = \sum_{k=1}^{N} \frac{R_k(t)}{x - x_k(t)} + \lambda(t)$$

Plugging u and it is derivatives u_t *and* u_x from equations 3, 4 and 5 we have:

$$\frac{\dot{R}_1}{\varepsilon} + \frac{R_1 \dot{x}_1}{\varepsilon^2} + G(\varepsilon) = -(\frac{R_1}{\varepsilon^2} + H_1(\varepsilon)) + \beta(\frac{R_1}{\varepsilon} + F(\varepsilon)) + \gamma$$

Equating to zero the coefficients of ε^m leads to the constraints:

$(i) \qquad \dot{x}_1 = -1$

$(ii) \qquad \dot{R}_1 = \beta R_1$

$(iii) \qquad G(0) = -H_1(0) + \beta F(0) + \gamma$

and

$(iv) \qquad G^{(n)}(0) = -H_1^{(n)}(0) + \beta F^{(n)}(0), \; n \geq 1$

From the first constraint, and since the RHS of the dynamical system does not depend on $x_1, x_2, ..., x_N$ we can conclude that N=1 without loss of generality i.e. we have only one pole. It's position is(–t+c). The phase portrait is parallel lines to the x-axis towards the left. From the second constraint we get $R_1 = e^{\beta t}$.

Hence the solution will be

$$u(x,t) = \frac{e^{\beta t}}{x + t - c} + \lambda(t), \; x + t - c \neq 0$$

where

$$\lambda(t) = \phi(t) - \frac{e^{\beta t}}{t - c}, \; t \neq c$$

From the constraints (iii) and (iv), $\dot{\lambda}(t) = \beta\lambda(t) + \gamma$. This imposes on ϕ the following condition,

$$\phi(t) = -\frac{\gamma}{\beta} - e^{\beta t}(\frac{1}{c - t} + k)$$

where k is an arbitrary constant.

2. Poles' dynamics of rational solution for the m-KdV equation

Let us investigate the possibility that the modified KdV equation

$$u_t = 6u^2 u_x - u_{xxx} \tag{11}$$

admits solutions which are, for all time, rational in x. Motion of poles of such solutions can then be studied as dynamical systems.

The m-KdV belongs to the equations of the class (10) when $\gamma_3 = 6$ and all the remaining coefficients vanish. According to theorem 2, its rational solution has simple poles. It must have the form

$$u(x,t) = \sum_{k=1}^{N} \frac{R_k(t)}{x - x_k(t)} + \lambda(t) \tag{12}$$

To find the constraints which the quantity x_k satisfy and the values of $R_k(t)$ and $\lambda(t)$ we use the analysis in section (2). Substitute (12) in equation (11) we get,

$$\frac{\dot{R}_1}{\varepsilon} + \frac{R_1 \dot{x}_1}{\varepsilon^2} + G(\varepsilon) = 6\left(\frac{R_1}{\varepsilon} + F(\varepsilon)\right)^2 \left(-\frac{R_1}{\varepsilon^2} + H_1(\varepsilon)\right) - \left(-6\frac{R_1}{\varepsilon^4} + H_3(\varepsilon)\right) \tag{13}$$

The functions F, G, H_1 and H_3 are defined in (6), (7) and (8); the coefficients of ε^{-4} leads to :

$$0 = 6R_1^3 - 6R_1 \quad \text{or} \quad R_1^2 = 1 \tag{14}$$

while the coefficients of ε^{-3} leads to $F(0) = 0$, i.e.

$$\sum_{k=2}^{N} \frac{R_k}{(x_1 - x_k)} + \lambda(t) = 0 \tag{15}$$

The dynamics of the poles is then determined by the vanishing of the coefficients of ε^{-2} as :

$$\dot{x}_1 = 6R_1(H_1(0) - 2F'(0))$$

and since $H_1(0) = F'(0)$ then,

$$\dot{x}_1 = \sum_{k=2}^{N} \frac{R_k}{(x_1 - x_k)^2} \tag{16}$$

DISCUSSION

We have investigated the problem of the assumption of rational solution to evolution equations. The partial fraction expansion theorem is utilized for the purpose of a unifying scheme for the rational solution.

We present the scheme as a simple algorithm, which gives the order of the poles, but it does not identify the number of poles.

The advantage in our approach is that it furnishes the sheme easily, as well as the construction does not depend on the available solutions. It can be applied to a wide class of PDEs.

Some of the results of KdV, Burgers and other equations [10,11] can be recovered from this paper. The family of equations (9) and (10) which are treated in [7] and include Burgers, Newell-Whitehead, KdV, m-KdV and other are studied here. All these equations have solutions of one kind of singularity (in his sence)[7], while in this paper their rational solutions have different orders of the poles.

We applied our approach to linear PDEs. We can indicate that all the linear equations can not have rational solutions (in our sence) except the first order ones. In this case imposing a condition on $u(x,t)$ in the origin gives the exact solution to the problem as well as the motion of it's poles. For nonlinear PDEs the approach is applied to the m-KdV.

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