ON THE WEIBULL STRESS-STRENGTH MODEL

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Abstract

In this paper we will discuss the divergence problem for $P(X < Y)$,where X and Y are two independent random variables follow Weibull distributions with different shape parameters and different scale parameters . The result which is reached here ,will show the heinous mistakes in many papers and theses.

I- INTRODUCTION

Inferences about $R = P(X < Y)$, where X and Y independent random variables, is very common in the reliability literature . For example , if Y is the strength of a component which is subject to a stress X , then R is a measure of system performance and arises in the context of mechanical reliability of a system. The system fails if and only if at any time the applied stress is greater than its strength.

It is well known that the probability density function of the weibull random variable w is,

$$f(w) = \frac{\alpha}{\sigma^{\alpha}} \; w^{\alpha-1} \exp\left\{-(w/\sigma)^{\alpha}\right\} \qquad , w > 0 \qquad -----(1)$$

Where $\alpha > 0$ is the shape parameter and $\sigma > 0$ is the scale parameter.

In this paper we will refer to the above by $W \sim WE(\alpha, \sigma)$,which is mean that the random variable W follow weibull distribution with parameters α and σ

Now, let X and Y be the stress and the strength random variables follow respectively $WE(\alpha_1, \sigma_1)$ and $WE(\alpha_2, \sigma_2)$, then by using power series and some mathematical steps , it can be easily shown that [6] ,

$R = P(X < Y)$

$$= 1 - \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)} \left(\frac{\sigma_2}{\sigma_1}\right)^{j\alpha_1} \Gamma\left(\frac{\alpha_1}{\alpha_2} j + 1\right) \qquad -----(2)$$

There are two important special cases of the formula in (2) above ,

(a) If the scale parameters are equal , $\theta = \sigma_1^{\alpha_1} = \sigma_2^{\alpha_2}$, one can write,

$\left(\frac{\sigma_2}{\sigma_1}\right)^{\alpha_1} = \frac{\left(\sigma_2^{\alpha_2}\right)^{\alpha_1/\alpha_2}}{\sigma_1^{\alpha_1}} = \theta^{\frac{\alpha_1}{\alpha_2}-1}$ and $b = j\left(\frac{\alpha_1}{\alpha_2} - 1\right) - 1$, then by using the fact that

$$\Gamma\left(\frac{\alpha_1}{\alpha_2} j + 1\right) = \left(\prod_{k=0}^{b} \left(\frac{\alpha_1}{\alpha_2} j - k\right)\right) \Gamma(j+1) \qquad -----(3)$$

can get ,

$$R = 1 - \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)} \left(\theta^{\frac{\alpha_1}{\alpha_2}-1}\right)^j \Gamma\left(\frac{\alpha_1}{\alpha_2} j + 1\right)$$

$$= 1 - \sum_{j=0}^{\infty} (-1)^j \left(\theta^{\frac{\alpha_1}{\alpha_2}-1}\right)^j \prod_{k=0}^{b} \left(\frac{\alpha_1}{\alpha_2} j - k\right) \qquad -----(4)$$

(b) If the shape parameters are equal , $\alpha_1 = \alpha_2$, one can get ,

$$R = 1 - \sum_{j=0}^{\infty} (-1)^j \left(\frac{\sigma_2}{\sigma_1}\right)^{j\alpha}$$

$$= 1 - \sum_{j=0}^{\infty} \left(\frac{-\sigma_2^{\alpha}}{\sigma_1^{\alpha}}\right)^j$$

Since the summation term of R above is geometric series ,then

$$R = 1 - \left(1 + \frac{\sigma_2^{\alpha}}{\sigma_1^{\alpha}}\right)$$

$$= \frac{\theta_2}{\theta_1 + \theta_2} \qquad -----(5)$$

Where $\theta_1 = \sigma_1^{\alpha}$ and $\theta_2 = \sigma_2^{\alpha}$.

The formula in (5) derived firstly in 1991 by McCool [7] from the other way.

II- THE DIVERGENCE PROBLEM

In this section we will check and discuss the convergence of the alternating series in (2) . At first ,let us rewrite the series in (2) as ,

$$R = \sum_{j=1}^{\infty} \frac{(-1)^j}{\Gamma(j+1)} \left(\frac{\sigma_2}{\sigma_1}\right)^{j\alpha_1} \Gamma\left(\frac{\alpha_1}{\alpha_2} j + 1\right) \qquad -----(6)$$

$$= \sum_{j=1}^{\infty} (-1)^j u_j$$

Where $u_j = \frac{1}{\Gamma(j+1)} \left(\frac{\sigma_2}{\sigma_1}\right)^{j\alpha_1} \Gamma\left(\frac{\alpha_1}{\alpha_2} j + 1\right)$

So, by using Leibniz test [4], which is based on satisfying the following two conditions,

(a) $u_{j+1} \le u_j \quad , \quad \forall j \in N$ and

(b) $\lim_{j\to\infty} u_j = 0$

To be the series in (6) converges, one can get,

(i) Let $\delta(j) = \frac{u_{j+1}}{u_j}$, so $\delta(j) \le 1$ is equivalent to $u_{j+1} \le u_j$, then by using (3) and

putting $d = (j+1)\left(\frac{\alpha_1}{\alpha_2} - 1\right) - 1$,

$$\delta(j) = \frac{(\sigma_2/\sigma_1)^{\alpha_1}\left\{\prod_{k=0}^{d}\left(\frac{\alpha_1}{\alpha_2}(j+1) - k\right)\right\}\Gamma(j+2)}{(j+1)\left\{\prod_{k=0}^{b}\left(\frac{\alpha_1}{\alpha_2}j - k\right)\right\}\Gamma(j+1)}$$

$$= \frac{j+2}{j+1}\left(\frac{\sigma_2}{\sigma_1}\right)^{\alpha_1}\prod_{k=1}^{\alpha_1/\alpha_2}\left(\frac{\alpha_1}{\alpha_2}j - k\right)$$

It is clear that $\delta(j)$ can not be less than or equal to one (or equivalently u_{j+1} can not be less than or equal to u_j) , except for some rare values of $\alpha_1 < \alpha_2$.

(ii) By using the fact in (3) , one can write,

$$u_j = \left(\frac{\sigma_2}{\sigma_1}\right)^{j\alpha_1}\prod_{k=0}^{b}\left(\frac{\alpha_1}{\alpha_2}j - k\right)$$

It is clear that also $\lim_{j\to\infty} u_j \ne 0$ except for some rare values of $\alpha_1 < \alpha_2$ and $\sigma_1 > \sigma_2$.

III- THE RESULT DISCUSSION

Some of authors, reviewers, members of discussions committees and supervisors committed egregious mistakes since they were thinking that the formula in (2) is correct for all values of α_1, α_2 , σ_1 and σ_2 .

Kotz, Lumelskii and Pensky [6] noticed this problem and stated that the series in (2) is convergent provided $\alpha_1 < \alpha_2$. They solved this problem for $\alpha_2 < \alpha_1$ case by calculating R as $R = P(X < Y) = 1 - P(X \geq Y)$, so they wrote,

$$R = \begin{cases} 1 - \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)} \left(\frac{\sigma_2}{\sigma_1}\right)^{j\alpha_1} \Gamma\left(\frac{\alpha_1}{\alpha_2} j + 1\right) & if \;\; \alpha_1 < \alpha_2 \\ \\ \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j+1)} \left(\frac{\sigma_1}{\sigma_2}\right)^{j\alpha_2} \Gamma\left(\frac{\alpha_2}{\alpha_1} j + 1\right) & if \;\; \alpha_1 < \alpha_2 \end{cases} \quad -----(7)$$

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