TRANSVERSE FREE VIBRATIONS OF TAPERED CANTILEVER BEAM

Nawal H. Al – Raheimy
College of Engineering,
Babylon University

ABSTRACT

Rayleigh and Rayleigh-Ritz represent the two approximate analytical methods which can be used in this paper in order to study the characteristics of free transverse vibrations of cantilever beam in the two cases. The first one is at tapered thickness and constant width while the second case is at tapered width and. The free end of the cantilever is sharp in both cases and is of different values of length. The study selects the value of thickness at clamped end (hc) and change the values of width at this end (wc) with thickness ratio which equals to ($\frac{wc}{hc} = 0.1 - 5$). The second case on the other hand, can select the value of width (wc) at clamped end then change the thickness (hc) at ratio ($\frac{hc}{wc} = 0.1 - 5$). Through out the results, it is shown that the cantilever beam at tapered thickness has natural frequency lesser than that of tapered width of the same length and at the same dimensions of clamped end, the natural frequency decrease with increasing the length of cantilever beam also with decreased the value of tapered thickness or tapered width according to this study. In case of different values of tapered width for the same length of beam and the same value of thickness at clamped end the cantilevers have the same value of frequency, also the cantilevers have constant frequency at tapered thickness for different values of ($\frac{wc}{hc}$), while in the case of tapered width the frequency increases with increasing the value ($\frac{hc}{wc}$) for the same (wc) and at the same length of cantilever beam.

Key words: Transverse vibrations, cantilever beam, tapered thickness, Tapered width.

الاهتزازات الحرة المستعرضة في العتبة الناتئة المتدرجة

الخلاصة

في هذا البحث تم استخدام طريقة رايلي وطريقة رايلي ريتز حيث إنهما يمثلان الطرق التحليلية التقريبية لدراسة خصائص الاهتزازات الحرة المستعرضة للعتبة الناتئة وفي حالتين، حيث ان الحالة الأولى عند سمك متدرج وعرض ثابت بينما الحالة الأخرى فإنها عند عرض متدرج وسمك ثابت و النهاية الحرة تكون حادة في كلتا الحالتين وعند أطوال مختلفة للعتبة الناتئة. تضمنت الدراسة اختيار قيمة معينة للسمك عند النهاية المثبتة (hc) والقيام بتغيير قيم العرض (wc) لهذه النهاية مع السمك لنسبة تساوي إلى) $\frac{wc}{hc} =$ (5 − 0.1. من الناحية الأخرى

في الحالة الثانية تم اختيار قيمة العرض (wc)عند النهاية المثبتة بعدئذ تغيير السمك (hc) عند النسبة). $\frac{hc}{wc}=$ (5 – 0.1تبين من خلال النتائج بان العتبة الناتئة ذات السمك المتدرج تمتلك تردد طبيعي اقل مما عليه من العتبة ذات العرض المتدرج عند نفس الطول المحدد وكذلك عند نفس الأبعاد للنهاية المثبتة وكذلك فان التردد الطبيعي يقل مع زيادة طول العتبة وكذلك عند تقليل قيمة السمك المتدرج أو العرض المتدرج وفقا للحالة المدروسة. عند قيم مختلفة للعرض المتدرج لنفس الطول للعتبة وعند نفس قيمة السمك للنهاية المثبتة فان العتبة تمتلك نفس التردد الطبيعي وكذلك فان العتبة تمتلك تردد طبيعي ثابت عند السمك المتدرج ولقيم مختلفة $(\frac{wc}{hc})$ ، بينما في حالة العرض المتدرج فان التردد يزداد مع زيادة القيمة $(\frac{hc}{wc})$ عند نفس (wc) وعند نفس الطول للعتبة.

LIST OF SYMPOLS

A	Arbitrary constant.
Ac	Area of cross section of beam at clamped end (m^2).
A(x)	Area of cross section of beam at section x (m^2).
C	Arbitrary constant.
C_1	Arbitrary constant.
C_2	Arbitrary constant.
E	Modulus of elasticity (N/m^2).
F(t)	Harmonic force (N).
hc	Thickness of beam at clamped end (cm).
h(x)	Thickness of beam at part of length (x) (cm).
L	Length of beam (m).
Ic	Second moment of area at clamped end (m^4).
I(x)	Second moment of area at part of length(x) (m^4).
K	Total stiffness of cantilever beam (N/m).
m	Total mass of beam per unit length (Kg/m).
m(x)	Mass of beam per part of length x (Kg/m).
T(t)	Kinetic energy (J).
t	Time (sec).
V(t)	Stain energy (J).
wc	width of beam at clamped end (cm).
w(x)	Width of beam at part of length(x) (cm).
x	Length of part of beam (m)
Y(x)	Transverse displacement mode.
$Y_r(x)$	Mode shape of order r.
$Y'(x)$	First derivative of displacement mode.
ρ	Density of material of beam (kg/m^3).
ω_1	Natural frequency of beam at mode 1 (rad/sec).
ω_r	Natural frequency of beam at mode r (rad/sec).

1. INTRODUCTION

Designs of engineering applications need to know the nature of vibration in structures in order to avoid the failure in the work and insure the stability of structures. Cantilevers are necessary in many engineering structures which used as resonator sensor have been already reported in the literature. **Wange, 1967** used analytical method to solution the differential equation of motion in term of hyper geometric series, the beam in rectangular cross section for any positive power variation of thickness. **Timoshenko, 1974** provided the necessary information for studied on geometry influences on free, forced, linear and/or nonlinear vibration of such cantilever. **Wright, 1982** presented study in an analytical solution for free vibration in term of power series by Frobenius method for tapered width & constant thickness was dedicated to beam of one end sharp. **Frieman and Kosmatka, 1992** presented an exact stiffness matrix of a nonuniform beam based on the flexibility–stiffness transformation approach. They included shear deformation effects in bending stiffness matrix of tapered members, so that it could be applied to Bernoulli–Euler and Timoshenko Beam. **Pepplewell, 1996** proposed for finding the free vibration of non uniform beam having material or cross sectional discontinuities, intermediate spring supports, or non classical end supports. **Ismail, 2000** In this study, transverse vibrations of a beam made of two materials and with a variable cross section were investigated. Dimensionless natural frequency values of the system were found by the Rayleigh-Ritz approach. Moreover, the energy amounts of the system accumulated per unit mass were calculated. The results were given in tables for comparison. **Maiti, 1999** studied beam of constant thickness with tapered width to find characteristics of vibrations in power series by Frobennius method. **Chan, 2003** his studies show that the stiffness of a tapered member will be reduced significantly due to the axial compression and shear deformation in certain cases where used Chebyshev polynomial approach to solve the second-order differential equation with variable coefficients. To develop a theoretical approach for second order inelastic analysis of steel frames of tapered members with slender web. **Caruntu 2005,** used analytical method to study the characteristics of beam in orthogonal polynomials**,** beam of parabolic thickness variation with particular boundary conditions. **Dumitru, 2007** This paper deals with transverse vibrations of nonuniform homogeneous beams and plates. Classes of beams and axisymmetrical circular plates whose boundary value problems of free transverse vibrations and free transverse axisymmetrical vibrations, respectively, can be reduced to an eigenvalue singular problem (singularities occur at both ends) of orthogonal polynomials, are reported. The geometry consists of parabolic thickness variation, with respect to the axial coordinate for beams, and with respect to the radius for plates. **Dumirtu, 2009** In this work, the fourth differential equation of motion factories into a pair of second order differential equations in term hypergeometric functions, and frequency equation resulting from cantilever boundary conditions, are reported. The exact of natural frequencies and exact mode shapes are found for sharp parabolic cantilevers by solving the frequency equation.

This paper, the approximate methods can be used to study the characteristics of transverse vibration of cantilever beam in tapered thickness in the first case and tapered width in the second case at sharp free end which has different dimensions ratio for clamped end in different values of length of cantilever beam.

2. THEORETICAL ANALYSIS

Derivation the equation of free transverse vibration of cantilever beam of linear tapered thickness of length L, with the following properties at section x; m(x) is the mass per unit length, A(x) is the cross-section area, I(x) is the moment of inertia and h(x) is the thickness, as in Fig.(1-a).

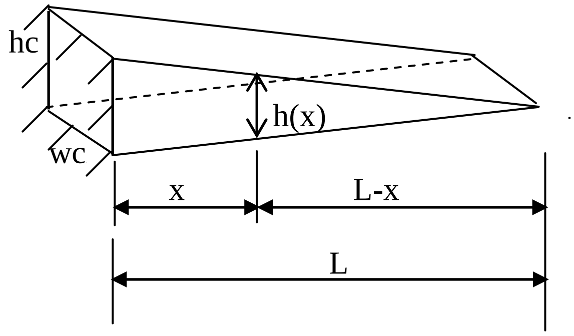

Fig.(1-a): Cantilever beam of tapered thickness

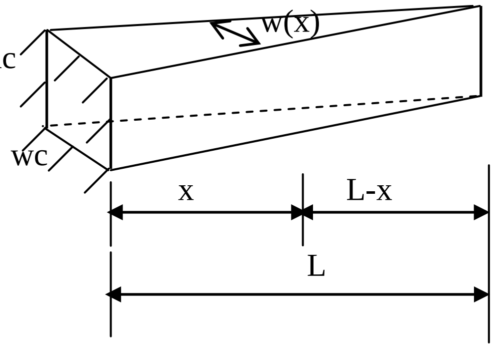

Fig.(1-b): Cantilever beam of tapered width

Fig.(1): Tapered cantilever beam

$$\frac{hc}{L} = \frac{h(x)}{L-x} \quad (1)$$

After simplified above relation yields:-

$$h(x) = h\ (1 - \frac{x}{L}) \quad (2)$$

In tapered thickness, A(x) =Ac(1-x/L),where Ac is represented the area of cross section at clamped end, therefore m(x)=m(1-x/L), where m is the mass per unit length and equal to ρ*Ac and $I(x) = Ic(1-x/L)^3$, where $Ic = wc*hc^3 / 12$.

Now the procedure of Rayeigh-Ritz can be application to derive the natural frequency for transverse motion of tapered thickness of cantilever beam. For the cantilever beam, guess for Yr(x) a function that equal zero at x=0 and at the free end, the deflection and slope must be not equal zero. Let us use the two simple terms approximately**.**

Yr(x) = C1 y1(x) + C2 y2(x) **[Benoraya, 1998].** (3)

$$\text{Yr(x)} = C1\left(\frac{x}{L}\right)^2 + C2\left(\frac{x}{L}\right)^3 \quad (4)$$

The strain energy of a bending beam is given by

$$V(t) = \frac{1}{2}\int_0^L E\,I(x)\left(\frac{\partial^2 Y}{\partial x^2}\right)^2 dx \quad (5)$$

And the kinetic energy is given by

$$T(t) = \frac{1}{2}\int_0^L m(x)\left(\frac{\partial^2 Y}{\partial t^2}\right) dx \quad (6)$$

can be written V(t) and T(t) as a function shown bellow after using the product solution, y(x,t) = Y(x) F(t)

$$V(t) = \frac{1}{2}F^2(t)\int_0^L E\,I(x)\left(Y''\right)^2 dx \quad (7)$$

$$T(t) = \frac{1}{2}\,(\ddot{F})^2\int_0^L m(x)\left(Y^2\right) dx \quad (8)$$

Substitute equation (4) or its derivatives in above equations (7&8) and F(t) is harmonic, say A cos ωt, and simplified them can obtain:-

$$V_{\max} = \frac{1}{2}A^2\int_0^L E\,I(x)\left(Y''\right)^2 dx \quad (9)$$

$$T\max = \frac{1}{2}\ A^2 \int_0^L\ m(x)\left(Y^2\right)dx \tag{10}$$

when derivative the above two equations w.r.t. C1&C2 will obtain

$$\frac{\partial V\max}{\partial c_1} = \sum_{j=1}^{2} k_{1j} c_j \tag{11}$$

$$\frac{\partial V\max}{\partial c_2} = \sum_{j=1}^{2} k_{2j} c_j \tag{12}$$

and

$$\frac{\partial T\max}{\partial c1} = \sum_{j=1}^{2} m_{1j} c_j \tag{13}$$

$$\frac{\partial T\max}{\partial c2} = \sum_{j=1}^{2} m_{1j} c_j \tag{14}$$

Where c1 & c2 are represented arbitrary constants of eigenvector.
and k_{11}=E Ic/L^3 , k_{12}= k_{21} = 3 E Ic/ 5L^3 , k_{22} = 3 E Ic/ 5L^3 , m_{11}=m L/30 , m_{12}=m_{21}= m L/42 and m_{22}=m L/56.

Now at the same procedure of tapered width of cantilever beam in Fig.(1-b) except w(x) =w(1-x/L) and I(x)= Ic(1-x/L), therefore the final relations as shown bellow :-
k_{11}=2E Ic/L^3 , k_{12}= k_{21} = 2 E Ic/ L^3 , k_{22} = 3 E Ic/ L^3 , m_{11}=m L/105 , m_{12}=m_{21}= m L/42 and m_{22}=m L/56. From all above relations can write in matrix form as

$$\begin{bmatrix} k_{11}-\omega^2\ m_{11} & k_{12}-\omega^2\ m_{12} \\ k_{12}-\omega^2\ m_{12} & k_{22}-\omega^2\ m_{22} \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{15}$$

or in general matrix notation as :

$$\left[[K]-\omega^2\,[M]\right]\{c\}=\{0\} \tag{16}$$

The evaluation of this determinant provides us with estimateion of the two natural frequency ω_1^2 and ω_2^2, since we have used a two term approximate solution resulting in a two degree of freedom approximate system.

Applying the Raliegh method after choosing a function which must accept the conditions of clamped – free ends, therefore we can select the function of the first mode as shown below.

$$Y(x) = C\left(1 - \cos\frac{\pi\,x}{2L}\right) \tag{17}$$ **[Benaroya, 1998]**

$$V = \frac{1}{2}\int_0^L E\,I(x)\left[\frac{d^2Y}{dx^2}\right]^2 dx \tag{18}$$

$$T = \frac{\omega^2}{2}\int_0^L m(x)\,[Y(x)]^2\,dx \tag{19}$$

In Ryliegh method the square of natural frequency equals to the ratio of potential energy to the kinetic energy, therefore substitutes the second derivative of equation (17) in equation (18) and substitute directly equation (17) in the equation (19) then simplify it in order to obtain the natural frequency of tapered cantilever beam into two cases as shown below :

For tapered thickness $\omega 1 = \frac{5.477}{L^2}\sqrt{\frac{E\,Ic}{m}}$ (20)

And for tapered width $\omega 1 = \frac{7.746}{L^2}\sqrt{\frac{E\,Ic}{m}}$ (21)

Finally in order to obtain the mode shape must return to equation (15) where $\begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$ represented the eigenvectors, can be found it after solving the 2*2 determinant $\left| \{K\} - \omega^2 \{M\} \right|$ =0 at natural frequency which vanished it, then we obtain the values $\begin{Bmatrix} 1 \\ \frac{c2}{c1} \end{Bmatrix}$ or $\begin{Bmatrix} \frac{c1}{c2} \\ 1 \end{Bmatrix}$ and substitute this values of eigenvector into equation (4) at different value of length (x) to find the displacement of mode shape .

3. RESULTS AND DISCUSSION:

. The trend of present-day structures is to be lightweight and to avoid resonant frequencies. Table (1) shows the comparison of the natural frequency of the first mode for transverse free vibrations of different length of beam between the Rayleigh method in the present work and the exact method in (Dumitru, 2009) at clamped-free boundary for tapered parabolic thickness and constant width. The Rayliegh method predicts the natural frequency higher than the exact method. Now the results based on the main properties of material E=200 Gpa, ρ=2700 Kg/m^3, figures (2 to 5) show that the first mode of vibration of tapered thickness as a function of the ratio (wc/hc) of cantilever beam obtained by the Rayleigh method and the Rayleigh – Ritz method for variation values of clamped thickness in different values of length of beam.. It is clearly seen that the Rayleigh method predict natural frequency is higher than Rayleigh-Ritz method. This figures show the natural frequencies remain constant as the ratio (wc/hc) increases at the same thickness (hc) and the same length of beam. It may be observed that the natural frequency increases with increasing the value of tapered thickness at clamped end and decreasing the length of beam. This behaviour can be explained the strain energy of structure increased with increasing the thickness and decreasing the length that cause increasing the stiffness of beam therefore caused increasing the natural frequency. In the other hand mathematically from eq.(20) the natural frequency changes as a function with (hc) that is appears as constant value in this function for this case therefore the natural frequency becomes constant. Figures (6 to 9) show the first mode of vibration of tapered width as a function of the ratio (hc/wc) of cantilever beam obtained by the Rayleigh method and the Rayleigh – Ritz method for variation values of clamped width in different value of length of beam. also Rayleigh method predict natural frequency higher than Rayleigh-Ritz method. It can be noted that the natural frequency increases with increasing the ratio of (hc/wc), increasing the value of tapered width and decreasing the length of beam because in the tapered width the strain energy increases greater than the increasing of kinetic energy that is caused increasing the natural frequency in this case. Figures (10 to 17) show the natural frequency of the first two modes as a function of length of beam for variable value of thickness (hc) and variable width (wc) for tapered thickness and tapered width respectively. Those figures show the natural frequency decreased with increasing the length of beam resultant of decreasing the stiffness of beam and increasing the mass with increasing the length., also at any value of ratio into two cases the difference between modes at length greater than (2 meters) becomes very small and the natural frequency remains constant because of the stiffness of structure decreased opposite increasing the mass and the structures become stability. Figures (18 to 21) show the effect of different values of length on the natural frequency where as at a variable ratio of (wc/hc) & (hc/wc) for tapered thickness and tapered width respectively, where in the tapered thickness the difference between lines remain constant at increasing the ratio (wc/hc) while in the tapered width the difference between curves increase with increasing the ratio (hc/wc) at different values of length because of the strain energy increase at high value of ratio (hc/wc) on the other hand the stiffness decreases at increasing the length of cantilever but the mass increase with increasing the length that is causes increasing the kinetic energy therefore the level of natural frequency decreases at different values of length. Figures (22&23) can be concluded that the main features of the mode shapes associated with the first two natural frequencies as a function of different length of beam for tapered thickness and tapered width. It can be noted also that the behavior of modes of vibrations of tapered thickness inverse that in the tapered width, the displacement of motion equal zero at clamped end, and maximum in the free end at both cases. In tapered thickness the difference between motion at different values of thickness (hc) is small while in the tapered width the different reaches to zero at different values of width (wc).

4.CONCLUSIONS:

To conclude the present study can be summarized as follows the cantilevers beam at tapered width have natural frequencies higher than that in the tapered thickness at the same length and at the same cross section area in the clamped end accordingly the frequencies increases with increasing the thickness or width in the tapered thickness or tapered width respectively at the same length of beam. The cantilever of tapered width in variable value of (wc) have the same frequency at constant value of (hc).

Table(1): Natural frequency (rad/sec) of transverse vibrations of the beam at tapered parabolic thickness.

hc (m)	wc/hc	Length (m)	R.M	Exact method	Difference δ %
0.05	1	1	416.6	401	3.7 %
=	=	2	105	101	3.8 %
=	=	3	46.3	44	3.45 %
0.1	=	1	833.2	804	3.5 %
=	=	2	208.3	201	3.5 %
=	=	3	92.58	89.33	3.5 %

δ=[(Rayliegh method– Exact method)/ Rayliegh method] *100%
Parabolic thickness; $h(x)=(1-x^2/L^2)$,

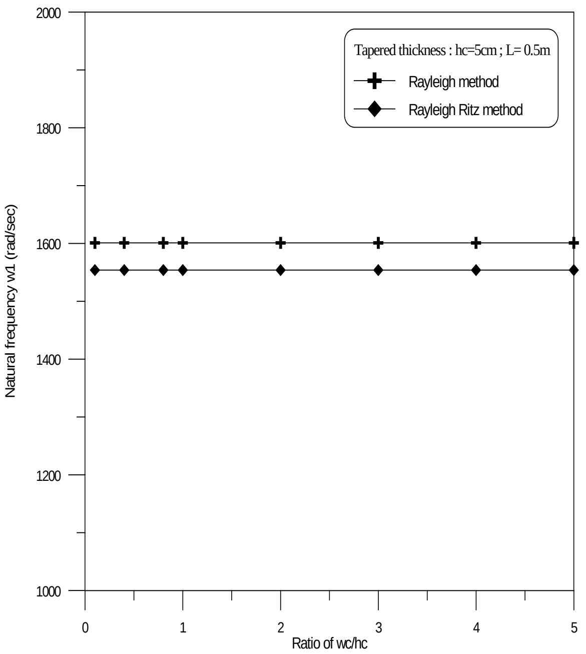

Fig.(2): Natural frequency of 1st. mode as a function of wc/hc for tapered thickness at half meter length for hc=5cm.

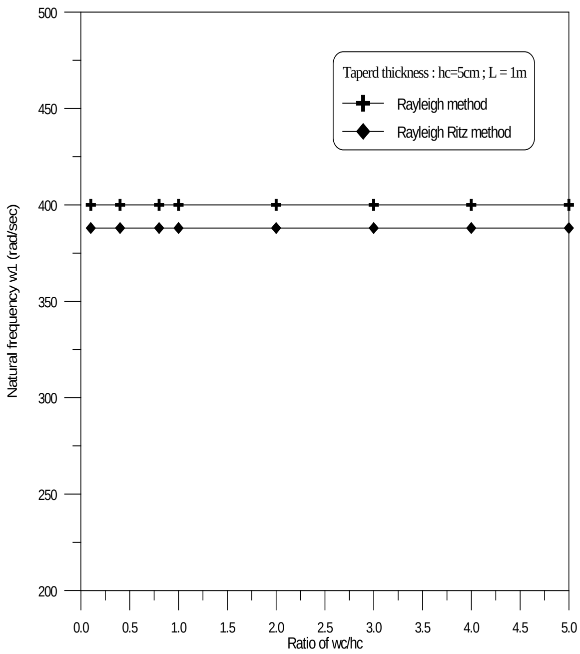

Fig.(3): Natural frequency of 1st. mode as a function of wc/hc for tapered thickness at one meter length for hc=5cm.

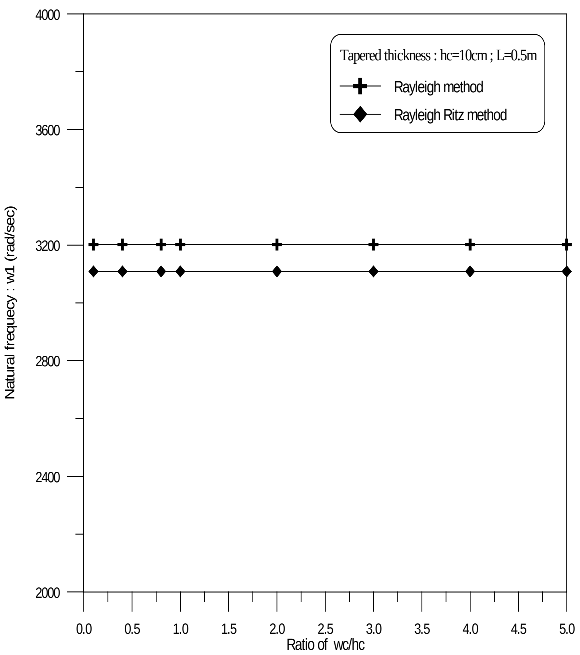

Fig.(4): Natural frequency of 1st. mode as a function of wc/hc for tapered thickness at half meter length for hc=10cm.

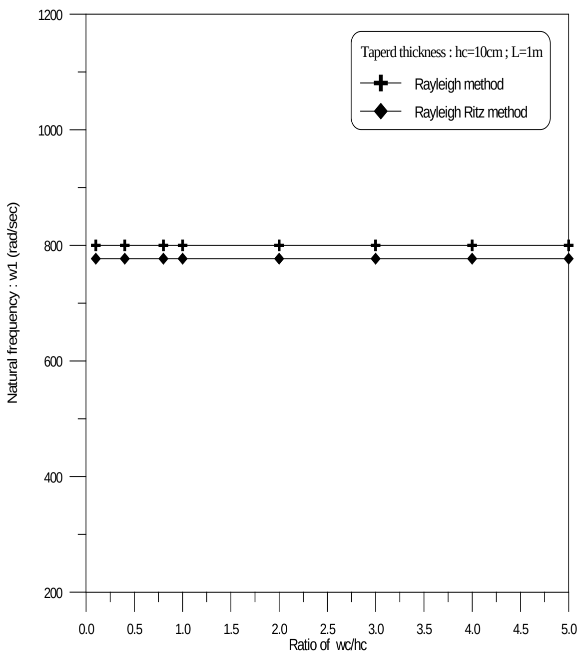

Fig.(5): Natural frequency of 1st. mode as a function of wc/hc for tapered thickness at half meter length for hc=10cm.

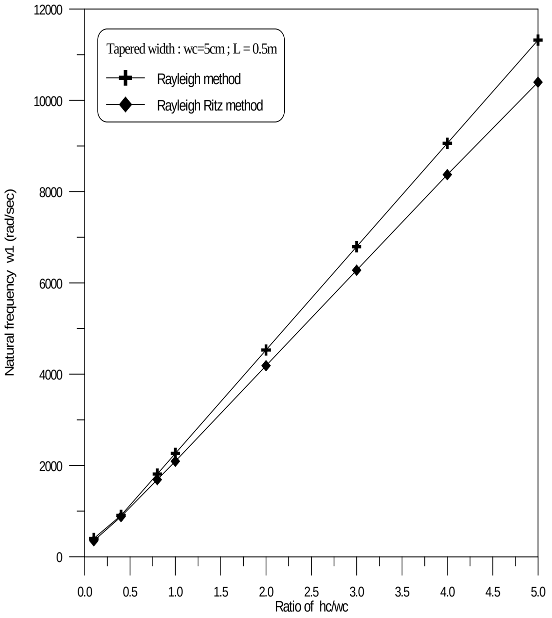

Fig.(6): Natural frequency of 1st. mode as a function of hc/wc for tapered width at half meter length for wc=5cm.

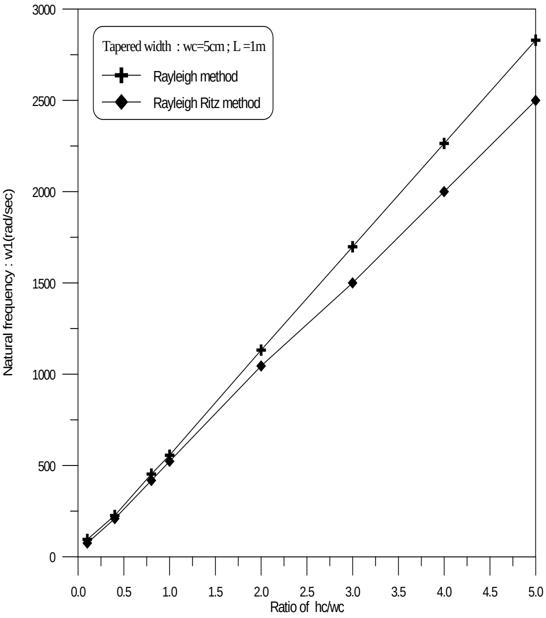

Fig.(7): Natural frequency of 1st. mode as a function of hc/wc for tapered width at one meter length for wc=5cm.

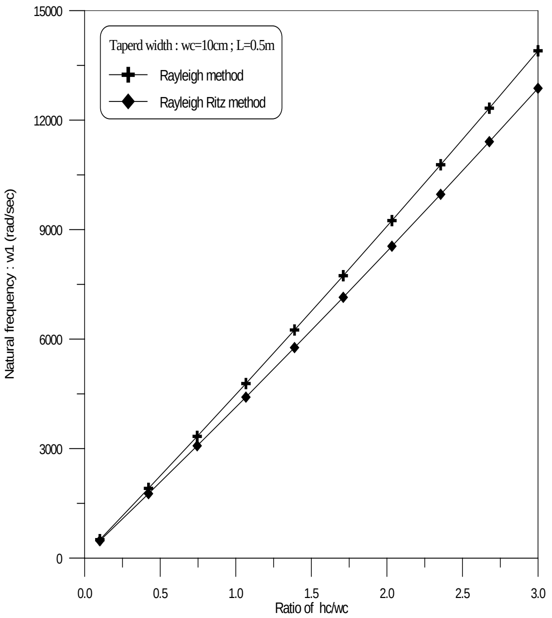

Fig.(8): Natural frequency of 1st. mode as a function of hc/wc for tapered width at half meter length for wc=10cm.

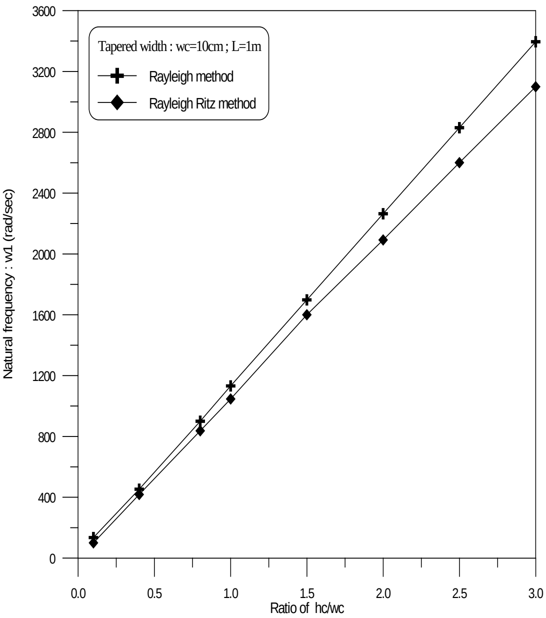

Fig.(9): Natural frequency of 1st. mode as a function of hc/wc for tapered width at one meter length for wc=10cm.

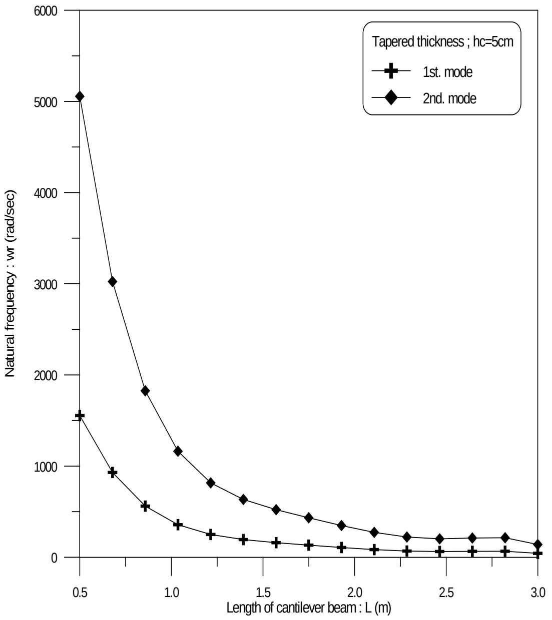

Fig.(10): Effect different length of C.B. on the first two modes of vibration at hc=5cm for any value of wc/hc using R-Ritz method.

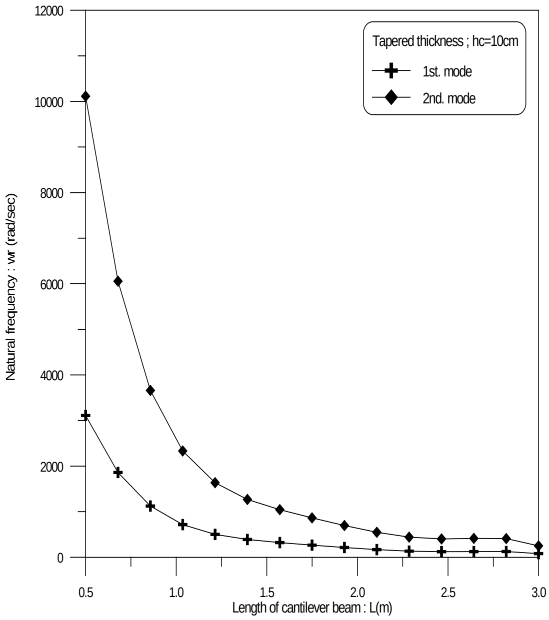

Fig.(11): Effect different length of C.B. on the first two modes of vibration at hc=10cm for any value of wc/hc using R-Ritz method.

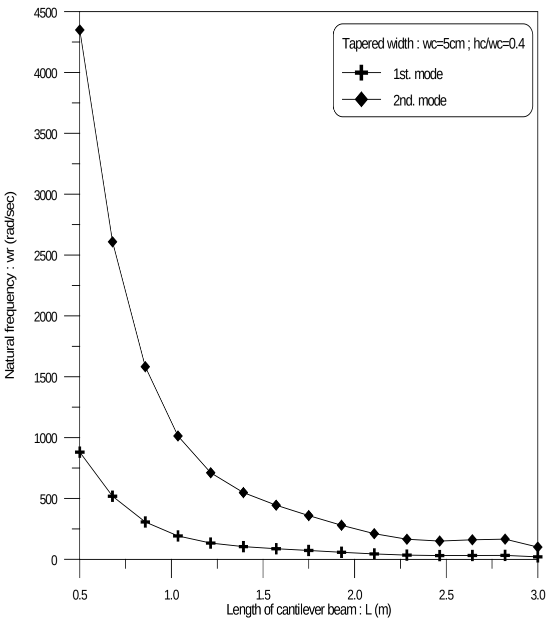

Fig.(12): Effect different length of C.B. on the first two modes of vibration at wc=5cm & hc/ wc=0.4 R-Ritz method.

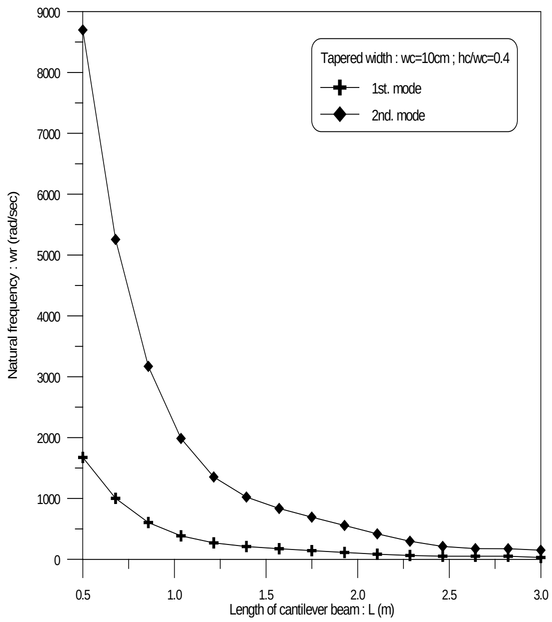

Fig.(13): Effect different length of C.B. on the first two modes of vibration at wc=10cm & hc/ wc=04 R-Ritz method.

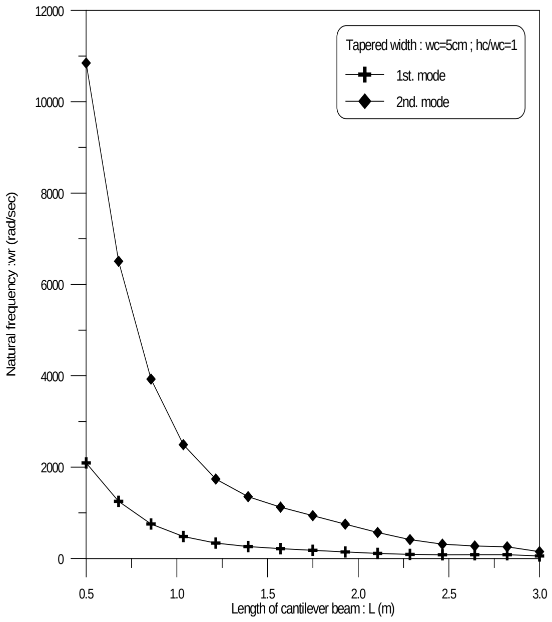

Fig.(14): Effect different length of C.B. on the first two modes of vibration at wc=5cm & hc/ wc=1.

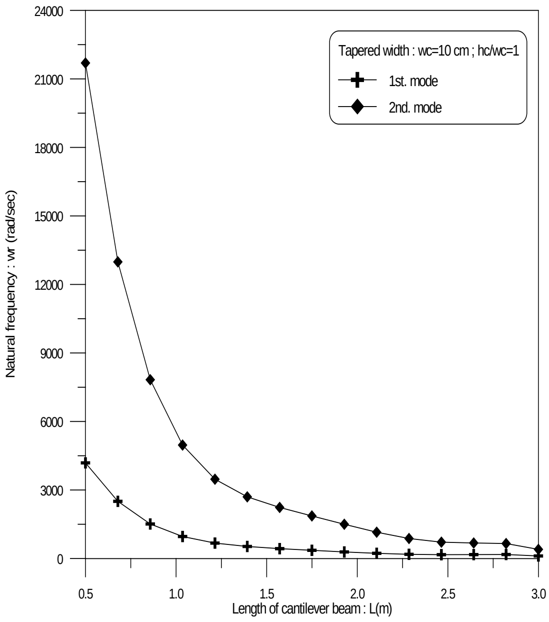

Fig.(15): Effect different length of C.B. on the first two modes of vibration at wc=10cm & hc/ wc=1.

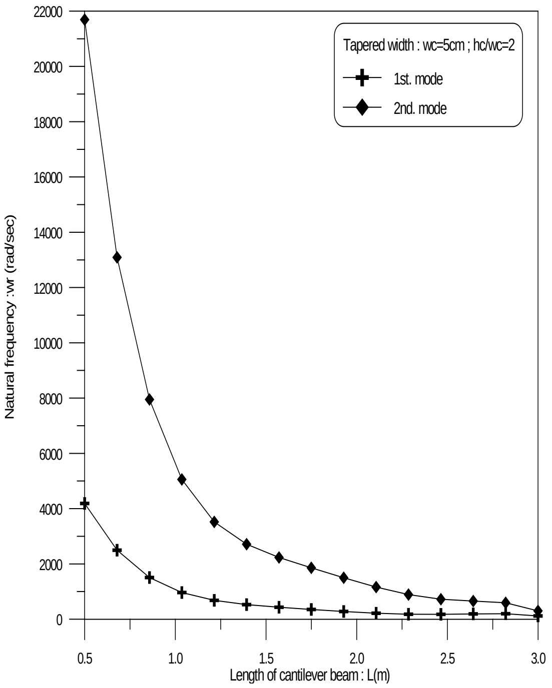

Fig.(16): Effect different length of C.B. on the first two modes of vibration at wc=5cm & hc/ wc=2.

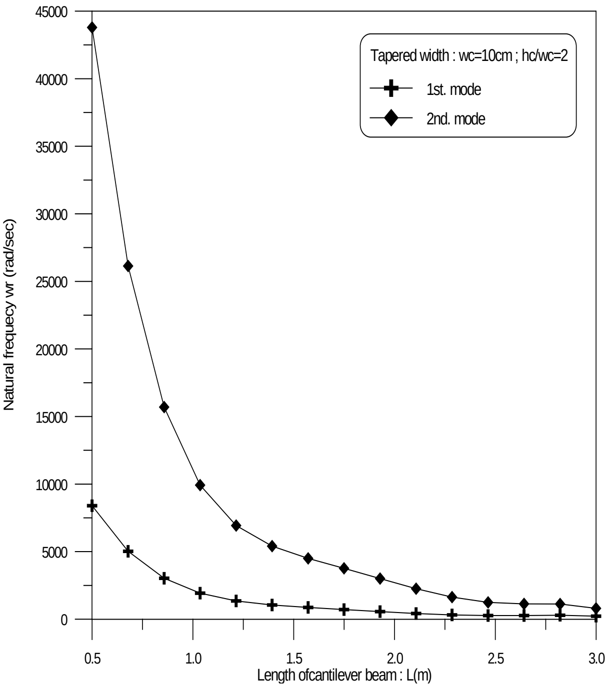

Fig.(17): Effect different length of C.B. on the first two modes of vibration at wc=10cm & hc/ wc=2.

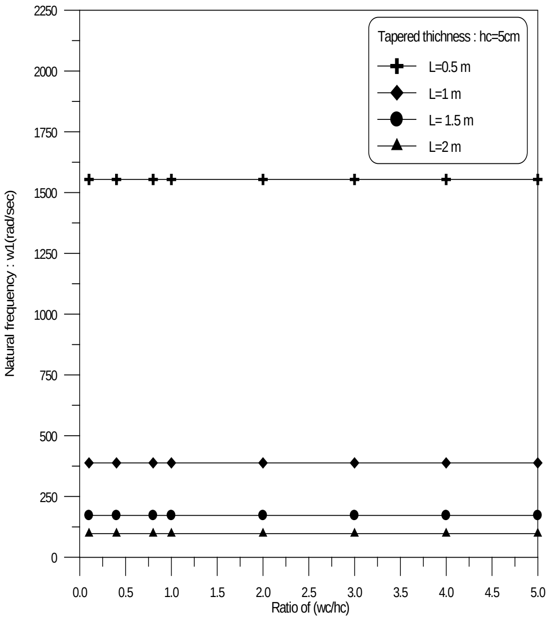

Fig. (18): Natural frequency as a function of thickness ratio of different value of length for 1st mode at hc=5cm using R-Ritz method

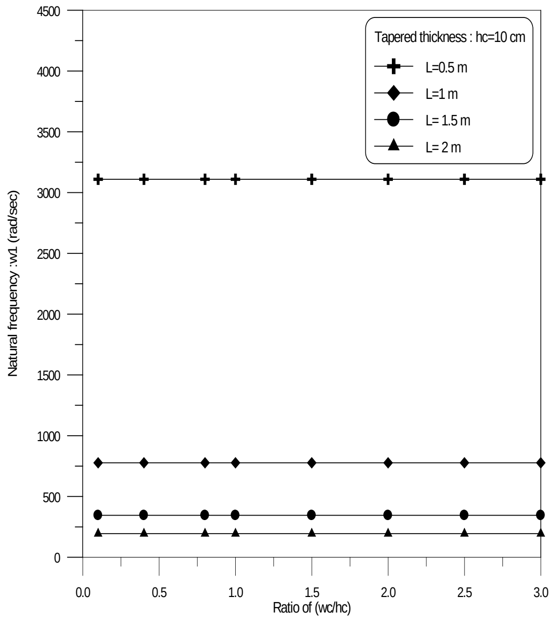

Fig. (19): Natural frequency as a function of thickness ratio of different value of length for1st mode at hc=10cm using R-Ritz method

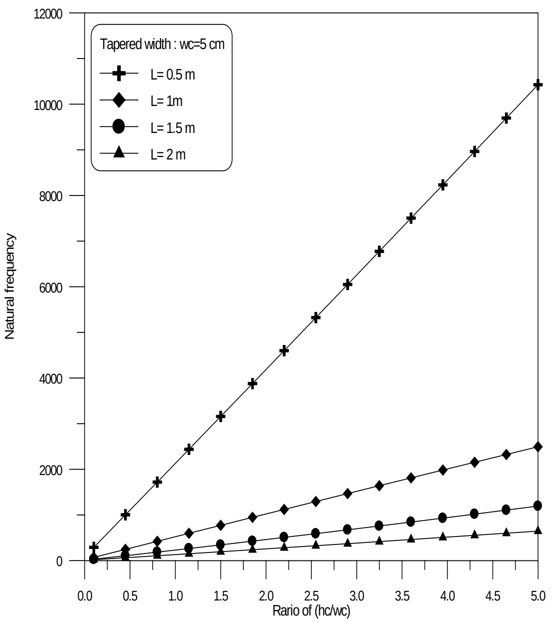

Fig. (20): Natural frequency as a function of thickness ratio of different value of length for 1st mode at wc=5cm using R-Ritz method

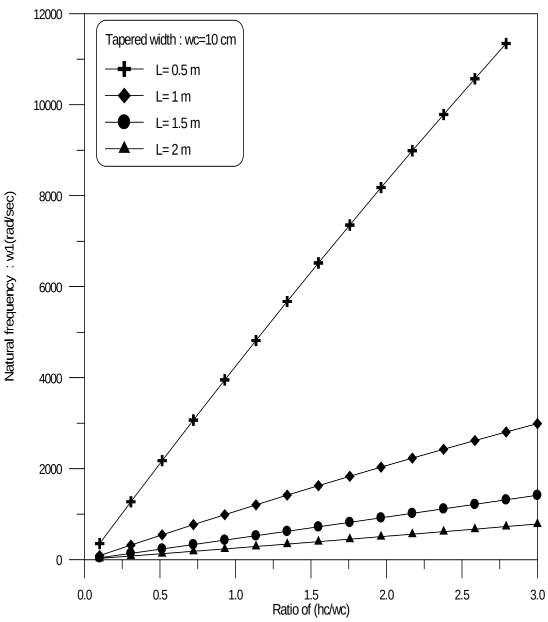

Fig. (21): Natural frequency as a function of thickness ratio of different value of length for 1st mode at wc=5cm using R-Ritz method

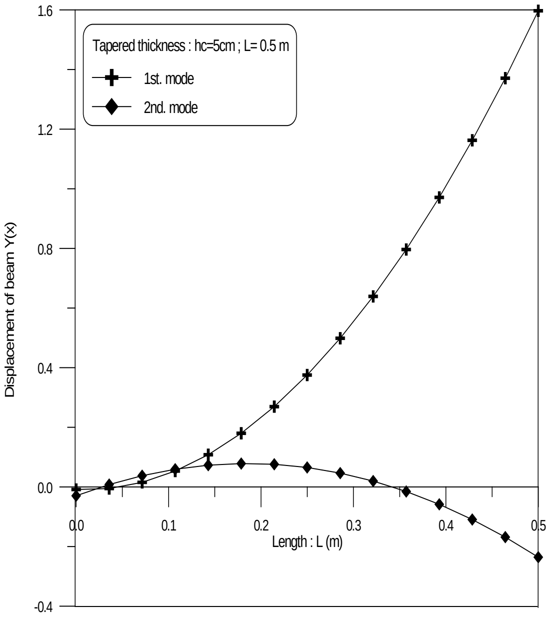

Fig. (22-a)

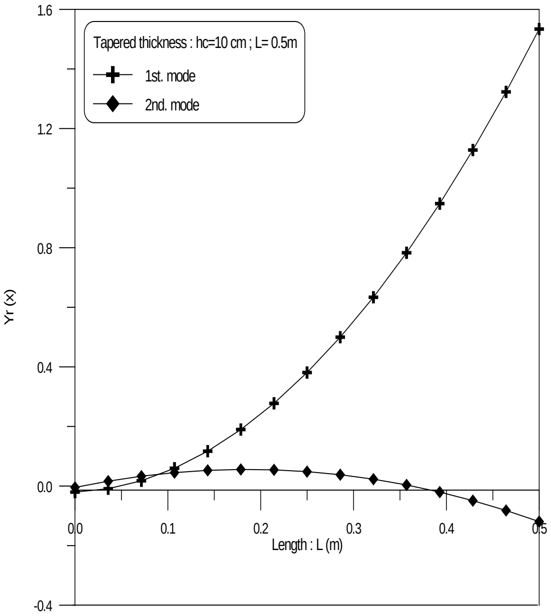

Fig. (22-b)

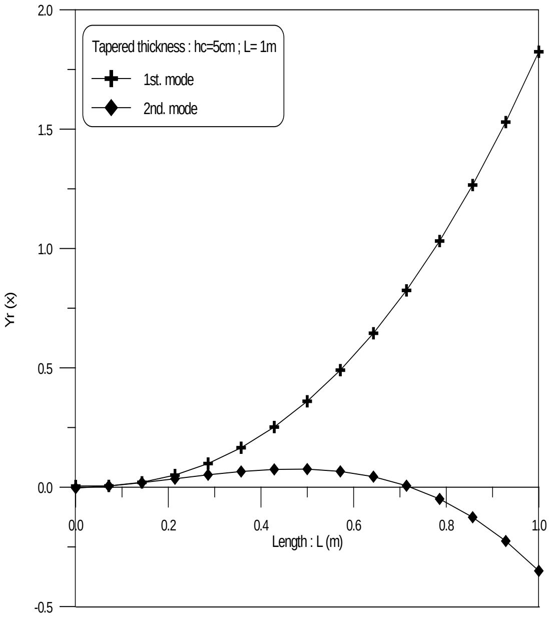

Fig. (22-c)

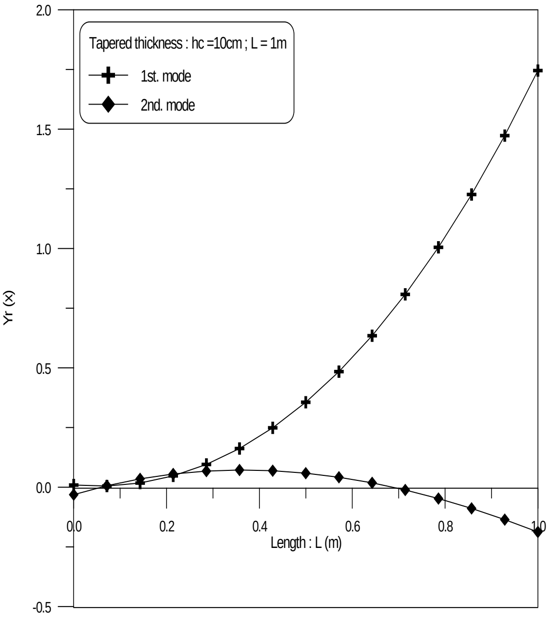

Fig. (22-d)

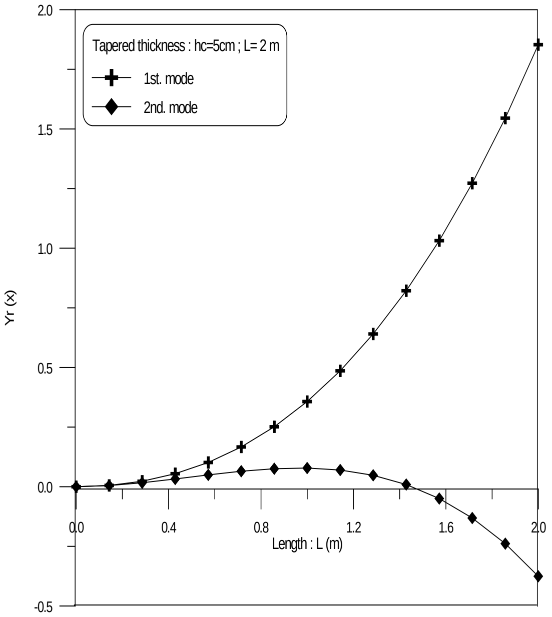

Fig. (22-e)

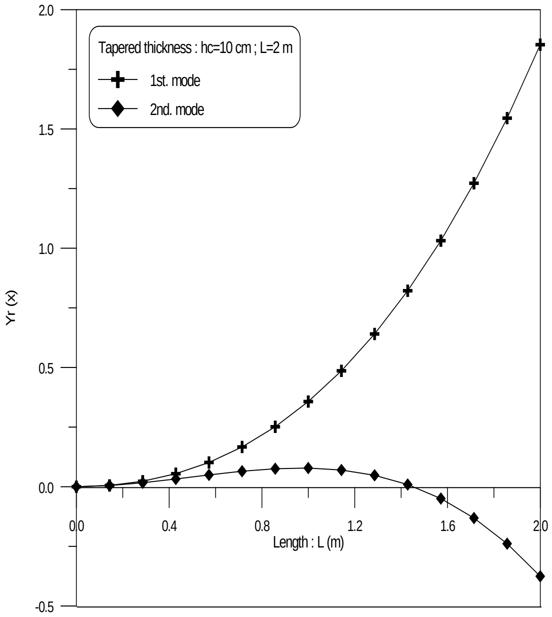

Fig. (22-f)

Fig.(22a–f): Mode shapes associated with the first two of natural frequency of tapered thickness of cantilever beam for different values of length

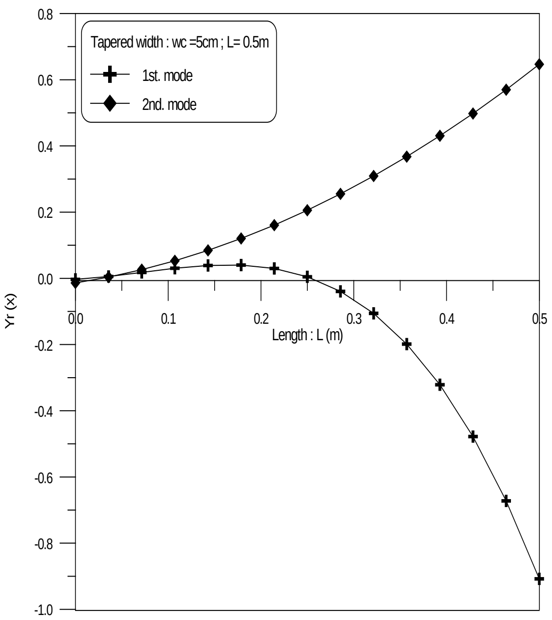

Fig. (23-a)

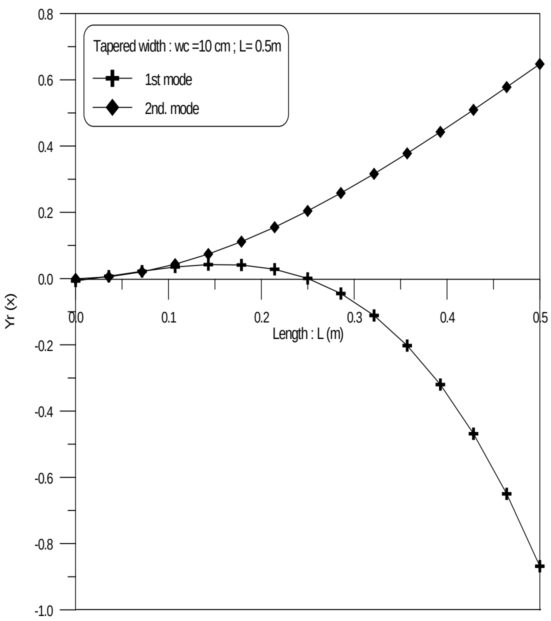

Fig. (23-b)

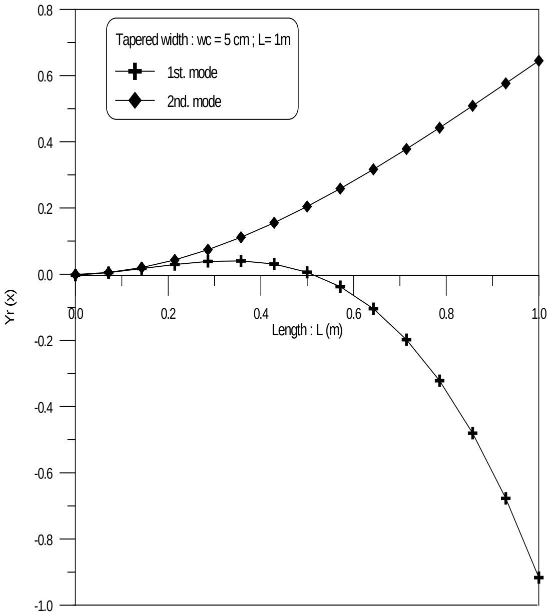

Fig. (23-c)

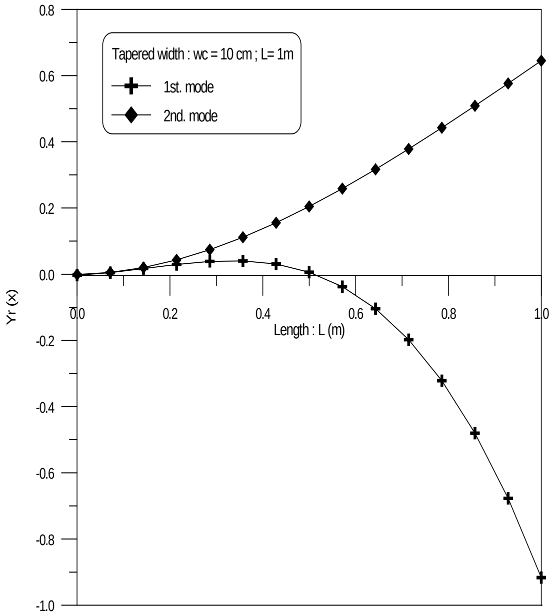

Fig. (23-d)

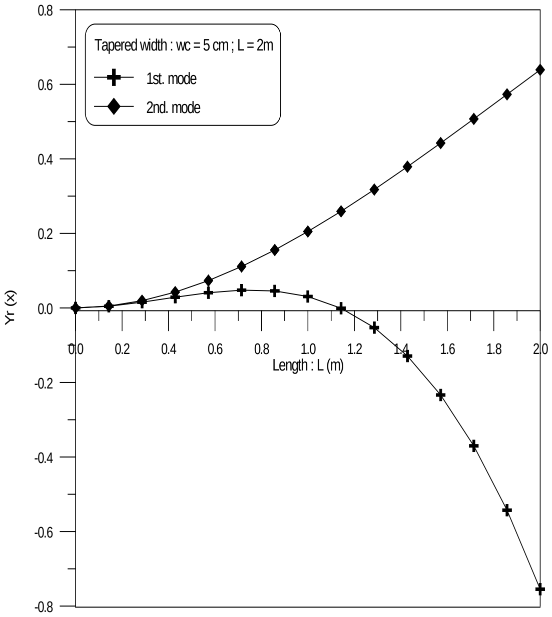

Fig. (23-e)

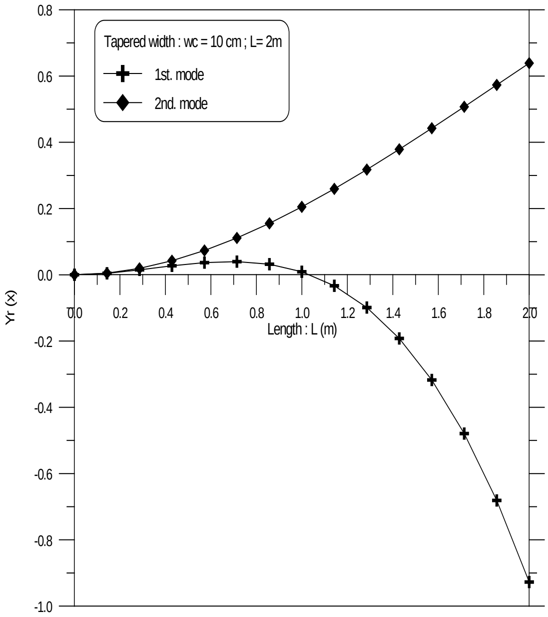

Fig. (23-f)

Fig.(23a–f): Mode shapes associated with the first two of natural frequency of tapered width of cantilever beam for different values of length

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