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On The Direct Sum Of Min(Max) - CS Modules

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Abstract.

In this paper, we study the direct sum of min (max)-CS modules. We show that the direct sum of min (max)-CS modules need not be min (max)-CS module. So we give many sufficient conditions under which the direct sum of min (max)-CS modules become min (max)-CS module

Key words: CS-module, min-CS module, max-CS module, uniform dimension (U–dim).

1- Introduction

Throughout this paper all rings R are commutative with identity and all R-modules are unitary. We write $A \leq M$ and $A \leq_e M$ to indicate that A is a submodule of M and A is an essential submodule of M, respectively.

A submodule N of M is called closed if N has no proper essential submodule extension; that is if $N \leq_e W$ for some $W \leq M$, then N=M. It is clear that M, (0) are closed submodules.

A submodule N of M is called maximal closed if whenever $N \subseteq W$ and W is a closed submodule, then W = M or N = W.

A submodule N of M is called minimal closed whenever $W \subseteq N$ and W is a closed submodule, then W = (0) or W = N.

About thirty years ago, M.Harda and B.Muller introduced the concept of extending module, where an R-module M is called an extending module (or, CS-module) if every submodule is an essential in a direct summand of M. Equivalently, M is extending if and only if every closed submodule is a direct summand, [1]. CS-modules have been studied by several authors such as N.V.Dung, D.V.Huyn, P.F.Smith and R. Wisbauer [2], S.H.Mohammed and B.J.Muller [3]. Many authors investigated extending relative to certain class of modules. Beside this, many generalizations of CS-modules are introduced see [4], [5].

S.H.Al-Hazmi in [6] introduce the concept of min (max)-CS module, where an R-module M is called min(max)-CS module if every minimal closed submodule (every maximal closed submodule of M with nonzero annihilator) is a direct summand of M.

Many basic properties of min(max)-CS modules are considered in [7]. In this paper we turn our attention to the direct sum of min(max)-CS modules, where we notice that the direct sum of min(max)-CS modules need not be min(max)-CS modules. However we give many sufficient and necessary conditions under which this property valid.

We start with the following:

1.1 Definition: [6]

An R-module M is called min-CS module if, every minimal closed submodule of M is a direct summand of M.

A ring R is called min-CS if it is min-CS R-module.

1.2 Definition: [6]

An R-module M is called max-CS module, if every maximal closed submodule of M with nonzero annihilator is a direct summand of M.

A ring R is max-CS if it is max-CS R-module.

We show that a direct sum of min-CS modules needs not to be min-CS module, as the following example shows:

Each of $\mathbb{Z}_2$ and $\mathbb{Z}_8$ are min-CS modules, but $M = \mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not min-CS.

We claim that the direct sum of max-CS modules need not be max-CS module, but we have no example to ensure this.

First we give the following**:**

1.2 Proposition:

Let M_1, M_2 be R-modules and let $M = M_1 \oplus M_2$ such that $annM_1 + annM_2 = R$. Then

1. M is a max-CS if M_1, M_2 are max-CS modules and the converse is true if M is not faithful.
2. M is a min-CS module if and only if M_1, M_2 are min-CS modules.

Proof:

(1) Let A be a maximal closed submodule of M with $annA \neq 0$. Then by the same proof of [8, Proposition 4.2, Ch.1], $A = B \oplus C$, where $B \leq M_1$ and $C \leq M_2$.
We claim that B and C are maximal closed submodules in M_1 and M_2 respectively.
First we shall prove that B and C are closed in M_1 and M_2 respectively.
Suppose there exists $B_1 \leq M_1$ such that $B \leq_e B_1$. Since $C \leq_e C$, we get $B \oplus C \leq_e B_1 \oplus C$ by [9, Proposition 5.20, p.75]. Hence $B \oplus C = B_1 \oplus C$, since $A = B \oplus C$ is a closed submodule of M.
So $B = B_1$. Thus B is a closed submodule of M_1.
Then by a similar way, C is closed in M_2.
To prove B is a maximal closed submodule of M_1.
Suppose there exists a closed submodule X of M_1 such that $B \leq X$.
Then $B \oplus C \leq X \oplus C$. But $X \oplus C$ is closed in $M_1 \oplus M_2$, by [10, Exc.15, p.20].
So $B \oplus C = X \oplus C$, since $B \oplus C$ is a maximal closed submodule of $M_1 \oplus M_2 = M$.
Hence $B = X$, and B is a maximal closed submodule of M_1.
By the same way we have C is a maximal closed submodule of M_2.
Now, $annA = annB \cap annC \neq 0$, and this implies $annB \neq 0$, $annC \neq 0$.
Thus there exists $W_1 \leq M_1$ and $W_2 \leq M_2$ such that $B \oplus W_1 = M_1$ and $C \oplus W_2 = M_2$.
Hence $(B \oplus W_1) \oplus (C \oplus W_2) = M_1 \oplus M_2 = M$.
Then $(B \oplus C) \oplus (W_1 \oplus W_2) = M$.
Thus $A \oplus (W_1 \oplus W_2) = M$, that is A is a direct summand of M.
Thus M is a max-CS module.
The converse follows by [7, Corollary 1.22].

(2) By a similar proof of (1), M is min-CS if M_1, M_2 are min-CS, and the converse follows by [7, Corollary 1.16].

Recall that an R-module M is called distributive if $A \cap (B + C) = (A \cap B) + (A \cap C)$ for all submodules A, B and C of M, [11].

Now, we give another condition under which, the direct sum of max-CS is max-CS.

1.3 Proposition:

Let M be a distributive R-module such that $M = M_1 \oplus M_2$ with $M_1, M_2 \leq M$. Then M is a max-CS if M_1 and M_2 are max-CS and the converse holds if M is not faithful.

Proof:

If M_1 and M_2 are max-CS modules.
To prove M is max-CS.
Let K be a maximal closed submodule of M with $annK \neq 0$.
Since M is a distributive, then $K = (K \cap M_1) \oplus (K \cap M_2)$.
Now, we claim that $K \cap M_1$ and $K \cap M_2$ are maximal closed in M_1 and M_2 respectively.
Suppose there exists a submodule B of M_1 such that $K \cap M_1 \leq_e B$. But $K \cap M_2 \leq_e K \cap M_2$.
Hence $K = (K \cap M_1) \oplus (K \cap M_2) \leq_e B \oplus (K \cap M_2)$ by Proposition [9, Proposition 5.20, p.75].
Since K is closed, it follows that
$(K \cap M_1) \oplus (K \cap M_2) = B \oplus (K \cap M_2)$.
So that $K \cap M_1 = B$. Thus $K \cap M_1$ is a closed submodule in M_1.
Similarly, $K \cap M_2$ is a closed submodule in M_2.
Now, we shall prove that $K \cap M_1$ is a maximal closed submodule in M_1.
Suppose there exists a closed submodule C in M_1 such that $(K \cap M_1) \leq C$.
Hence $K = (K \cap M_1) \oplus (K \cap M_2) \leq C \oplus (K \cap M_2)$, but $C \oplus (K \cap M_2)$ closed in M, by [10, Exc.15, p.20].
Thus $K = (K \cap M_1) \oplus (K \cap M_2) = C \oplus (K \cap M_2)$, since K is a maximal closed submodule in M.
This implies $K \cap M_1 = C$ and so $K \cap M_1$ is a maximal closed submodule of M_1.
Similarly, $K \cap M_2$ is a maximal closed submodule of M_2.
Moreover, $annK \neq 0$ and $ann K = ann(K \cap M_1) \cap ann(K \cap M_2) \neq 0$.
Hence $ann(K \cap M_1) \neq 0$ and $ann(K \cap M_2) \neq 0$.

On the other hand, M_1 and M_2 are max-CS modules. Then $K \cap M_1$ is a direct summand in M_1 and $K \cap M_2$ is a direct summand in M_2.
So that $M_1 = (K \cap M_1) \oplus A_1$ for some $A_1 \leq M_1$, and $M_2 = (K \cap M_2) \oplus B_1$ for some $B_1 \leq M_2$.
Then

$$M=M_1\oplus M_2=[(K \cap M_1)\oplus A_1]\oplus[(K \cap M_2)\oplus B_1]$$
$$= [(K\cap M_1)\oplus(K \cap M_2)]\oplus(A_1 \oplus B_1)$$
$$= K \oplus (A_1 \oplus B_1)$$

So that K is a direct summand of M.
Hence M is a max-CS module.

The converse follows by [7, Corollary 1.22].

We obtain a similar result for min-CS modules but first we prove the following lemma.

1.4 Lemma:

Let M be an R-module, such that $M = M_1 \oplus M_2$, where $M_1, M_2 \leq M$ and K is a minimal closed submodule of M.
Then either $K \cap M_1 = 0$ or $K \cap M_2 = 0$.

Proof:

Suppose $K \cap M_2 \neq 0$, then $K \cap M_2 \subseteq K$. Then there exists a closed submodule H of K such that $K \cap M_2 \leq_e H$, by [10, Exc.13, p.20].
But $M_1 \leq_e M_1$, so $(K \cap M_2) \cap M_1 \leq_e H \cap M_1$, by [10, Proposition 1.1, p.16-17], and hence $0 \leq_e H \cap M_1$. Thus $H \cap M_1 = 0$.
On the other hand, H is closed in K and K closed in M. Then H is closed in M, by [10, Proposition 1.5, p.18].
But K is minimal closed in M, so that K = H.
Hence $K \cap M_1 = 0$.

1.5 Proposition:

Let M be an R-module such that $M=M_1\oplus M_2$ with $M_1 \leq M$, $M_2 \leq M$ and M is distributive. Then M_1 and M_2 are min-CS if and only if M is min-CS.

Proof:

($\Rightarrow$) To prove M is a min-CS. Let K be a minimal closed submodule of M.
Then $K = (K \cap M_1) \oplus (K \cap M_2)$, since M is a distributive.
So that by Lemma 1.4, either $K \cap M_1 = 0$ or $K \cap M_2 = 0$.
Assume $K \cap M_2=0$, then $K = K \cap M_1$ and hence $K \subseteq M_1$.
But $K \subseteq M_1$ and K is minimal closed in M implies K is minimal closed in M_1.

To explain this:
If $K \leq_e L$ and $L \leq M_1$. Then $L \leq M$ and $K \leq_e L \leq M$.
So K = L since K is closed in M. Thus K is closed in M_1.
Now, assume there exists a closed submodule H in M_1 such that $H \leq K$.
Since H is closed in M_1, then H is closed in M, by [10, Proposition 1.5, p.18]. But K is a minimal closed in M. So that H = K.
Hence K is a minimal closed submodule in M_1.
But M_1 is a min-CS module.
So K is a direct summand in M_1.
Therefore, there exists a submodule W of M_1 such that $K \oplus W = M_1$.

$$M = M_1 \oplus M_2 = (K \oplus W) \oplus M_2 = K \oplus (W \oplus M_2)$$

So that K is a direct summand in M.
Hence M is a min-CS module.
($\Leftarrow$) It follows by [7, Corollary 1.16].

1.6 Proposition:

Let M be an R-module. $M = \underset{i\in I}{\oplus} M_i$, M_i is a max-CS module for each $i \in I$, such that every maximal (minimal) closed submodule in M is fully invariant. Then M is a max-CS (min-CS) module.

Proof:

Let S be a maximal closed submodule of M, and let $\pi_i: M \longrightarrow M_i$ be the natural projection on M_i for each $i \in I$.
Let $x\in S$, then $x = \sum_{i\in I} m_i$, $m_i\in M_i$ and $\pi_i(x) = m_i$.
Since S is a maximal closed in M, then by our assumption, S is fully invariant and hence $\pi_i(S) \subseteq S \cap M_i$.
So, $\pi_i(x) = m_i \in S \cap M_i$ and hence $x\in\oplus(S\cap M_i)$.
Thus $S \subseteq \oplus(S \cap M_i)$.
But $\oplus(S \cap M_i) \subseteq S$, therefore $S = \oplus(S \cap M_i)$.
Since $(S \cap M_i)$ is summand in S, then $(S \cap M_i)$ is closed in S, by [10, Exc.3, p.19]. But S is closed in M, so $(S \cap M_i)$ is closed in M, by [10, Proposition 1.5, p.18].
We claim that $(S\cap M_i)$ is maximal closed in M_i.
To prove our assertion
Suppose there exists a closed submodule B_i in M_i (for each $i \in I$), such that $S \cap M_i \subseteq B_i$. So $\underset{i\in I}{\oplus}(S\cap M_i) \subseteq \underset{i\in I}{\oplus} B_i$ .Hence, $S \subseteq \underset{i\in I}{\oplus} B_i$.

But $\underset{i\in I}{\oplus}B_i$ is closed in M = $\underset{i\in I}{\oplus}M_i$, by [10, Exc.15, p.20], and S is a maximal closed, then $S\subseteq\underset{i\in I}{\oplus}B_i$, that is $\underset{i\in I}{\oplus}(S\cap M_i)=\underset{i\in I}{\oplus}B_i$.
So $S \cap M_i = B_i$ for each $i \in I$; thus $S \cap M_i$ is a maximal closed submodule of M_i. Which implies that $S \cap M_i$ is a direct summand of M_i, since M_i is a max-CS module. Then
S = $\underset{i\in I}{\oplus}(S\cap M_i)$ is a direct summand of
$\underset{i\in I}{\oplus}M_i$ = M.
Hence M is a max-CS module.

Let S be a minimal closed submodule of M. By the same argument of the first paragraph, for each $i \in I$, $S \cap M_i$ is a closed submodule of M.
But $S\cap M_i\subseteq S$, hence $S\cap M_i = S$; that is $S \subseteq M_i$ for any $i \in I$.
On the other hand, S is a closed in M and $S \subseteq M_i$ implies that S is a closed in M_i for all $i \in I$.
To prove S is minimal closed in M_i, for all $i\in I$.
Suppose there exists $B_i \subseteq S$, $i \in I$, B_i is a closed in M_i.
But M_i is a closed in M, hence B_i is a closed in M, by [10, Proposition 1.5, p.18].
Thus B_i = S and so S is minimal closed in M_i.
Hence there exists $H_i \leq M_i$ such that $S\oplus H_i=M_i$ for all $i \in I$. Thus $\oplus(S \oplus H_i)=\underset{i\in I}{\oplus}M_i$ =M, $H_i\leq M_i$. Hence $S \oplus (\oplus_{i\in I} H_i) = M$.
So S is a direct summand of M and M is a min-SC module.

By [10, Exc.13, p.20], for any $N \leq M$, there exists a closed submodule H of M such that $N \leq_e H$. Sometimes, H is called a closure of N, See [4].

The following definition is given in [4]:

1.7 Definition: [4]

An R-module M is called an UC-module if each of its submodules has a unique closure in M.

1.8 Remark:

Every uniform R-module M is UC-module, since (0) and M are the only closed submodules of M. Thus (0) is the unique closure of 0 and for each $N \leq M$, $N \neq (0)$ M is the unique closure of N.

Now, we give the following result:

1.9 Proposition:

Let $M = M_1 \oplus M_2$ be an UC-R-module, with $M_1, M_2 \leq M$. Then M is a min-CS if and only if M_1 and M_2 are min-CS modules.

Proof:

($\Rightarrow$) It is clear by [7, Corollary 1.16].
($\Leftarrow$) Suppose that both of M_1 and M_2 is min-CS module.
To prove M is a min-CS module.
Let N be a min-closed submodule of M.
Then by Lemma 1.4 $N \cap M_1=0$ or $N \cap M_2=0$.
Suppose $N \cap M_2 \neq 0$.
Now, N is a closed submodule in M and M_2 is a closed submodule of M, therefore by [4, Lemma 1.3 (ii), p.70] we have $N \cap M_2$ is closed in M_2.
Now, we claim that $N \cap M_2$ is a minimal closed submodule in M_2.
For this, suppose there exists a closed submodule U of M_2 such that $U \subseteq N \cap M_2$. But M_2 closed in M, since every direct summand is closed, by [10, Exc.3, p.19]. Therefore, U is closed submodule in M, by [10, Proposition 1.5, p.18].
But $U \leq N \cap M_2 \leq N$, and N is a minimal closed submodule in M.
So that U = N thus $N \cap M_2$ = U = N.
Thus N is a submodule of M_2, since U = N.
Hence N is a minimal closed submodule in M_2.
But M_2 is a min-CS module. Therefore N is a direct summand of M_2.
Hence, there exists a submodule K of M_2 such that $N \oplus K = M_2$. It follows that $(N\oplus K)\oplus M_1= M_2 + M_1 = M$, so $N \oplus (K \oplus M_1) = M$.
Therefore N is a direct summand of M.
Hence M is a min-CS module.

1.10 Remark:

The condition M is UC cannot be dropped from the Proposition 1.9, as the following example shows:

1.11 Example:

Let M be the $\mathbb{Z}$-module $M = \mathbb{Z}_8\oplus\mathbb{Z}_2$.
M is not UC-module, since there exists $N = <(\bar{4},\bar{0})> = \{(\bar{4},\bar{0}),(\bar{0},\bar{0})\}$, $N \leq_e \mathbb{Z}_8\oplus(0)$ where $\mathbb{Z}_8\oplus(0)$ is a closed submodule of M, also $N \leq_e W = <(\bar{1},\bar{1})> = \{(\bar{1},\bar{1}),(\bar{2},\bar{0}),(\bar{3},\bar{1}),(\bar{4},\bar{0}),(\bar{5},\bar{1}),(\bar{6},\bar{0}),(\bar{7},\bar{1}),(\bar{0},\bar{0})\}$.
However as we noticed before M is not a min-CS module and each of $\mathbb{Z}_8$, $\mathbb{Z}_2$ is a min-CS module.

1.12 Definition: [2, p.22]

An R-module M is called ∑-*extending* (respectively, finitely or countably ∑-extending) if every (finite, countable) direct sum of copies of M is extending.

Similarly Al-Hazmi in [6, p.25] defined the following:

An R-module M is ∑-min-CS (finitely, countably ∑ min-CS) if every (finite, countable) direct sums of copies of M is min-CS. There exists a commutative ring R such that R (as R-module) is CS, but R is not finitely ∑-CS. [6]

Also there exists a regular ring R such that R is countably ∑-CS but R is not ∑-CS, [6].

Similarly, we can define the following:

An R-module M is said to be ∑-max-CS (respectively finitely or countably ∑-max-CS) if every (respectively finitely or countably) direct sum of copies of M is max-CS.

Now, we can give the following result:

1.13 Proposition:

A ring R is min (max)-CS if and only if R is finitely ∑ min (max)-CS.

Proof:

(⇒) If R is min-CS. To prove $R \oplus R \oplus \ldots \oplus R$ for n-times is min-CS.

With out loss of generality, we can take n = 2.

Let K be a minimal closed ideal in $R \oplus R$.

Then $K = I \oplus J$ for some ideals I and J of R.

It follows that I and J are minimal closed ideals of R. To see this:

Assume $I \leq_e H$, H is an ideal of R.

Then $I \oplus J \leq_e H \oplus J$ by [9, Proposition 5.20, p.75]. So that $I \oplus J = H \oplus J$, since $I \oplus J$ is closed in $R \oplus R$. Then I = H.

Hence I is closed ideal in R.

Similarly, we can prove J is closed in R.

Now, to prove I is a minimal closed ideal in R, let K be a closed ideal in R such that $K \subseteq I$.

Then $K \oplus J \subseteq I \oplus J$.

Hence $K \oplus J = I \oplus J$, since $I \oplus J$ is a minimal closed ideal in $R \oplus R$.

Thus K=I, and I is a minimal closed ideal in R.

Similarly, J is a minimal closed ideal in R.

Hence each of I and J is a direct summand of R, since R is a min-CS ring.

Thus $I \oplus I_1 = R$, $J \oplus J_1 = R$ for some ideals I_1, J_1 of R. It follows that:

$$R \oplus R = (I \oplus I_1) \oplus (J \oplus J_1) = (I \oplus J) \oplus (I_1 \oplus J_1)$$

Therefore $I \oplus J$ is a direct summand of $R \oplus R$.

Hence $R \oplus R$ is a min-CS ring.

(⇐) It is clear by [7, Corollary 1.16].

A similar proof can be given for finitely ∑-max-CS.

By similar proof we have:

1.14 Remark:

For any ring R, R is CS if and only if R is finitely ∑-CS.

1.15 Example:

Every uniform ring is CS and hence min-CS and max-CS. Then $\underbrace{R \oplus R \oplus \ldots \oplus R}_{n-\text{times}}$ is CS and hence min-CS and max-CS.

As particular case each of $\mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$, $\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \ldots \oplus \mathbb{Z}_8$ is CS and hence min-CS and max-CS.

1.16 Corollary:

Let R be a nonsingular ring. Then the following statements are equivalent:

(1) R is min-CS.
(2) R is max-CS.
(3) R is finitely ∑-min-CS.
(4) R is finitely ∑-max-CS.

Proof:

(1) ⇔ (2): It follows by [7, Theorem 1.33].
(1) ⇔ (3): It follows by Proposition 1.13
(2) ⇔ (4): It follows by Proposition 1.13.

1.17 Proposition:

Let R be a ring, then the following are equivalent:

(1) R is ∑ min-CS (∑ max-CS).
(2) Every projective R-module M is min (max)-CS, such that ann$M \neq 0$.

Proof:

(⇒) Suppose that R is ∑ min (max)-CS.

To prove every projective R-module is a min-CS (max-CS). Let M be a projective R-module. Then by [12, Theorem 5.4.1, p.120] there exists a free R-module F and an epimorphism $f: F \longrightarrow M$, but F is free so $F = \bigoplus_{i \in I} R$ for some index I.

Now, consider the following short exact sequence:

$$0 \longrightarrow \ker f \xrightarrow{\ i\ } \underset{I}{\oplus} R \xrightarrow{\ f\ } M \longrightarrow 0$$

where i is the inclusion map.
Since M is projective, then the sequence splits.
Thus $\underset{I}{\oplus} R = \ker f \oplus M$.
But $\underset{I}{\oplus} R$ is min-CS (max-CS).
Therefore, by [7, Corollary (2.1.16)] M is min-CS (max-CS).
(⇐) By [12, Theorem 5.3.4(b),p.118], $\underset{I}{\oplus} R$ is projective. So $\underset{I}{\oplus} R$ is min (max)-CS by condition (2).

By using a similar argument we can prove the following:

1.18 Proposition:
Let R be a ring. The following statements are equivalent:

(1) R is finitely ∑ min (max)-CS.
(2) Every finitely generated projective R-module is min(max)-CS.

Recall that, an R-module M has a uniform dimension (briefly U-dim) if M does not contain an infinite direct sum of nonzero submodules, [2, p.40].

Also Goodearl, see [10, p.79, p.86], gave the name finite dimensional module for module with finite uniform dimension.

First we give the details of the proof of the following lemma.

1.19 Lemma: [2, Lemma 7.7, p.58]
Let M be a min-CS R-module, and let K be a closed submodule of M with finite uniform dimension. Then K is a direct summand of M.

Proof:
Since K has a finite uniform dimension. Then there exists a submodule U of K such that U is a uniform closed of K, (since by the definition of a finite uniform dimension K has a uniform submodule say N and by [10, Exc.13, p.20] there exists a closed submodule U of K such that $N \leq_e U$. It follows that U is uniform) ; that is U is a minimal closed submodule of K, by [7, Lemma 1.6].
Thus U closed in K and K closed in M.
Therefore U closed in M, by [10, Proposition 1.5, p.18].
But U is a minimal closed in K, so U is a minimal closed in M.
Hence U is a direct summand of M, since M is a min-CS module.
Hence M = U ⊕ U' for some U' ≤ M.
Thus K = K ∩ (U ⊕ U'), which implies that K = U ⊕ (K ∩ U') by modular law.
So K ∩ U' is a closed submodule of K, by [10, Exc.3, p.19].
Again, since K is closed in M, we get (K ∩ U') closed in M, by [10, Proposition 1.5, p.18].
Now, we shall use induction to prove K is a direct summand.
Since U − dim(K ∩ U') ≤ U − dim(K), by Theorem [10, p.87].
Hence K ∩ U' is a direct summand of M.
Therefore, M=(K ∩ U') ⊕ W for some W≤M.
So that U' = U' ∩ [(K ∩ U') ⊕ W].
Then U' = (K ∩ U') ⊕ (U' ∩ W), by modular law.
But M = U ⊕ U', so that
M = U ⊕ [(K ∩ U') ⊕ (U' ∩ W)]
 = [U ⊕ (K ∩ U')] ⊕ (U' ∩ W)
That is M = K ⊕ (U' ∩ W).
Thus K is a direct summand of M.

1.20 Corollary: [2, Corollary 7.8, p.59], [6, Lemma 2.1.4, p.32]
Let M be an R-module with finite uniform dimension. Then M is CS if and only if M is a min-CS.

Proof:
(⇒) It is clear.
(⇐) Suppose M is a min-CS module.
To prove M is a CS-module.
Let K be a closed submodule of M.
Then by [2,5-10, p.41] U–dim(K)≤U–dim(M).
But M has finite uniform dimension, so K has finite uniform dimension.
Hence K is a direct summand of M by Lemma 1.19. Therefore, M is a CS-module.

Recall that for a faithful multiplication R-module M, M has finite uniform dimension if and only if R has finite uniform dimension, see [13, Theorem 2.15].

Hence we get the following:

1.21 Corollary:
Let M be a faithful finitely generated multiplication over a finite uniform dimension R. Then the following are equivalent:

(1) M is a min-CS module.
(2) R is a CS-ring.

(3) R is a min-CS ring.
(4) M is a CS-module.

Proof:

(1) ⇔ **(3)** It follows by [7, Proposition 1.30].
(2) ⇔ **(3)** It follows by Corollary 1.20.
(1) ⇔ **(4)** It follows by [13, Theorem 2.15, and Corollary 2.2.19].

1.22 Corollary:

Let M be a min-CS module. M has a finite uniform dimension if and only if M is a finite direct sum of uniform submodules.

Proof:

(⇒) If M is min-CS and M has a finite uniform dimension, then M is CS by Corollary 1.20.
Then M is a finite direct sum of uniform submodules, by [14, Lemma 6.43, p.222].
(⇐) Suppose M has no finite uniform dimension, so for each $V \leq_e M$, V is not a finite direct sum of uniform submodules, by [10, p.87]. But this is a contradiction, since $M \leq_e M$ and M is a finite direct sum of uniform submodules.
Hence M has a finite uniform dimension.

Recall that, an R-module M is indecomposable if the only direct sum decompositions M = A ⊕ B are those in which either A = 0 or B = 0, [10, p.4].

1.23 Corollary:

Let M be an indecomposable R-module. Then M is CS if and only if M is min-CS.

Proof:

It follows directly by Corollary 1.20, since every indecomposable module has a finite uniform dimension, see [2, p.40].

Recall that, a Goldie ring is a ring with finite uniform dimension such that the annihilator ideal satisfying the ACC, [10, p.97].

1.24 Corollary:

Let R be a Goldi ring. Then R is CS if and only if R is min-CS.

Proof:

It follows by Corollary 1.20.

1.25 Corollary:

Let R be semiprime Goldie ring. Then the following statements are equivalent:

(1) R is finitely ∑ min-CS.
(2) R is finitely ∑-CS.
(3) R is CS.
(4) R is min-CS.
(5) R is max-CS.
(6) R is finitely ∑ max-CS.

Proof:

(3) ⇔ **(4)**: It follows by Corollary 1.24.
(1) ⇔ **(4)**: It follows by Proposition 1.13.
(2) ⇔ **(3)**: It follows by Remark 1.14.
(4) ⇔ **(5)**: Since R is semiprime then R is nonsingular, by [10, Corollary 3.32, p.97]. Hence the result follows by [7, Theorem 1.33].
(5) ⇔ **(6)**: It follows by Proposition 1.13.

We end this paper by the following examples:

1.26 Examples:

(1) The ℤ-module $M = \mathbb{Z}_p \oplus Q$, where p is a prime number. M is not a CS-module. See [4, Example 1.2, p.70]. But M has a finite uniform dimension; therefore M is not a min-CS module, by Corollary 1.20.

(2) The ring $R = \begin{bmatrix} \square_2 & \square_2 \\ 0 & \square \end{bmatrix}$, R_R is not CS by [5, p.1248]. But R has a finite uniform dimension, th`us R is not min-CS ring by Corollary 1.20.

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حول المجموع المباشر لأصغر (أعظم) مقاسات التوسع

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المستخلص:

في هذا البحث نقوم بدراسة المجموع المباشر لاصغر (اعظم) مقاسات التوسع. نلاحظ انه المجموع المباشر لاصغر (اعظم) مقاس توسع ليس بالضرورة ان يكون اصغر (اعظم) مقاس توسع. لهذا نقوم باعطاء العديد من الشروط الكافية التي تحقق ان المجموع المباشر لاصغر (اعظم) مقاسات التوسع هي اصغر (اعظم) مقاسات التوسع.

الكلمات المفتاحية: مقاس توسع، أصغر مقاس توسع، أعظم مقاس توسع، بعد منتظم.