On A Subclass Of *P*- Valent Functions With Negative Coefficsients Defined By Integral Operator By Applying Fourier Series I

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Abstract :

In this paper , we introduced a new subclass $S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$ which consists of analytic and $p-$valent functions with negative coefficients in the unit disk defined by integral operator . We obtain coefficient estimates and some results including applications of Fourier series .

Mathematics Subject Classification: 30C45.

Keywords and Phrases :

$p-$valent Functions , Integral Operator , Fourier Series .

1.Introduction

Let $T_p(p\in IN)$ denote the class of functions of the form :

$$f(z)=z^p+\sum_{n=1}^{\infty} a_{p+n}z^{p+n} \quad , \quad (1)$$

which are analytic and $p-$valent in the unit disk $U=\{z: z\in C \text{ and } |z|<1\}$.

Let S_p denote the subclass of T_p consisting of functions of the form :

$$f(z)=z^p-\sum_{n=1}^{\infty} |a_{p+n}|z^{p+n} \quad , \quad (2)$$

Definition 1:

For $-1\le A<1$ and $0\le\delta<p$, $0<\lambda\le 1$, $0\le\mu<1$, $0<t\le 1$, $\beta\ge 0$, $\theta\ge -1$, a function $f\in T_p$ is said to be in the class $T_p^*(A,\delta,\lambda,\mu,t,\beta,\theta)$ if and only if

$$\left|\frac{(\lambda+\mu)\left[z(Q_\theta^\beta f(z))'-p(Q_\theta^\beta f(z))\right]}{tz(Q_\theta^\beta f(z))'-(Ap+(t-A)\delta)(Q_\theta^\beta f(z))}\right|<1 \ ,$$

(3)

where (Q_θ^β) is the generalized Jan – Kim – Srivastava integral operator [2] defined by

$$(Q_\theta^\beta f(z))=\frac{\Gamma(\beta+\theta+p)}{z\Gamma(\beta p)\Gamma(\theta+p)}\int_0^z t^{\theta-1}(1-t)^{\beta-1}f(t)dt$$

$$=z^p-\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)|a_{p+n}|z^{p+n} \ ,$$

(4)

where

$$\Psi(n,\beta,\theta,p)=\frac{\Gamma(\beta+\theta+p)\Gamma(\theta+n)}{\Gamma(\theta+p)\Gamma(\beta+\theta+n)} \ ,$$

(5)

and for $\beta=0$ we have $Q_\theta^0 f(z)=f(z)$. Let

$$S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)=T_p^*(A,\delta,\lambda,\mu,t,\beta,\theta)\cap S_p$$

(6)

We note that class $S_p^*(-1,0,1,0,1,0,0)$ was studied by Geol and Sohi [1] . The class $S_1^*(-1,\delta,1,0,1,0,0)$ was studied by

Silvarman [4].The class $S_p^*(-A,0,1,0,1,0,0)$ was studied by Pashkokeva and Vasilev [3] .

2. Coefficient Estimates

In the following theorem , we obtain the coefficient estimates for the class $S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$.

Theorem 1:
A function $f(z)$ defined by (2) be in the class $S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$ if and only if

$$\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)\left[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)\right]\left|a_{p+n}\right|\le(t-A)(p-\delta)$$

(7)

The result is sharp .

Proof :
Let (7) holds true and $|z|=1$. Then

$$\left|(\lambda+\mu)\left[z(Q_\theta^\beta f(z))'-p(Q_\theta^\beta f(z))\right]\right|$$
$$-\left|tz(Q_\theta^\beta f(z))'-(Ap+(t-A)\delta)(Q_\theta^\beta f(z))\right|$$
$$\left|\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)n(\lambda+\mu)\left|a_{p+n}\right|z^{p+n}\right|$$
$$-\left|(t-A)(p-\delta)z^p-\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)\left[t(n+p-\delta)-A(p-\delta)\right]\left|a_{p+n}\right|z^{p+n}\right|$$
$$\le\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)\left[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)\right]\left|a_{p+n}\right|-(t-A)(p-\delta)\le0$$

Hence by the principle of maximum modulus $f(z)\in S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$.

Conversely , suppose that

$$\left|\frac{(\lambda+\mu)\left[z(Q_\theta^\beta f(z))'-p(Q_\theta^\beta f(z))\right]}{tz(Q_\theta^\beta f(z))'-(Ap+(t-A)\delta)(Q_\theta^\beta f(z))}\right|$$

$$=\left|\frac{\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)n(\lambda+\mu)\left|a_{p+n}\right|z^{p+n}}{(t-A)(p-\delta)z^p-\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)\left[t(n+p-\delta)-A(p-\delta)\right]\left|a_{p+n}\right|z^{p+n}}\right|<1$$

Using the fact that $|\mathrm{Re}(z)|\le|z|$ for any z , we have

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)n(\lambda+\mu)\left|a_{p+n}\right|z^{p+n}}{(t-A)(p-\delta)z^p-\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)\left[t(n+p-\delta)-A(p-\delta)\right]\left|a_{p+n}\right|z^{p+n}}\right\}<1$$

(8)

Choose vales of z on the real axis so that $\frac{z(Q_\theta^\beta f(z))'}{Q_\theta^\beta f(z)}$ is real . Upon clearing the denominator in (8) and letting $z\to1^-$ through real values we obtain

$$\sum_{n=1}^{\infty}\Psi(n,\beta,\theta,p)\left[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)\right]\left|a_{p+n}\right|\le(t-A)(p-\delta)$$

The function

$$f(z)=z^p-\frac{(t-A)(p-\delta)}{\Psi(n,\beta,\theta,p)\left[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)\right]}z^{p+n}$$

(9)

is extermal function .

Corollary 1:
If $f(z)\in S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$,then

$$\left|a_{p+n}\right|\le\frac{(t-A)(p-\delta)}{\Psi(n,\beta,\theta,p)\left[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)\right]}$$

(10)

Theorem 2:

Let $f(z) \in S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$. Then the integral operator

$$F_k(z) = (1-k)z^p + kp\int_0^z \frac{f(u)}{u}du \ ,$$

$(k \ge 0, z \in U)$

(11)

is also in $S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$ if $0 \le k \le \frac{1}{p}$

Proof :

By virtue of (11) it follows from (1) that

$$F_k(z) = (1-k)z^p + kp\int_0^z \left(\frac{u^p - \sum_{n=1}^{\infty} |a_{p+n}| u^{p+n}}{u} \right) du$$

$$= z^p - \sum_{n=1}^{\infty} \gamma(n,k,p)|a_{p+n}| z^{p+n}$$

(12)

where $\gamma(n, k, p) = \frac{kp}{n}$.

But

$$\sum_{n=1}^{\infty} \Psi(n,\beta,\theta,p)[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)]\gamma(n,k,p)|a_{p+n}|$$

$$\sum_{n=1}^{\infty} \Psi(n,\beta,\theta,p)[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)]\left|\frac{kp}{n}\right||a_{p+n}|$$

$$\le \sum_{n=1}^{\infty} \Psi(n,\beta,\theta,p)[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)]|kp||a_{p+n}|$$

.

Since $|kp| < 1$ and by (7) last expression is less than or equal to $(t-A)(p-\delta)$, so the proof is complete .

The Fourier series is defined by the form :

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

such that a_0, a_n, b_n are constant . But $a_0 = a_n = 0$, we get

$$z^p(1 - z^n f(x)) = z^p + \sum_{n=1}^{\infty} b_n \sin(nx) z^{p+n}$$

(14)

Let the function $Y(z,x)$ be defined by the form :

$$Y(z,x) = f(z) * z^p(1 - z^n f(x))$$

$$= z^p + \sum_{n=1}^{\infty} |a_{p+n}| b_n \sin(nx) z^{p+n} \quad (15)$$

where $f(z)$ defined by (2) .

In the next theorem, we show that the function $Y(z,x)$ be in the class $S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$.

Theorem 4:

Let $f(z) \in S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$ be defined by (2). Then the function $Y(z,x)$ defined by (12) be in the $S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$, if $b_n \le 1$, $-2\pi \le nx \le 2\pi$.

Proof :

To prove the function $Y(z,x) \in S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$, we must to show that

$$\sum_{n=1}^{\infty} \Psi(n,\beta,\theta,p)[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)]$$

$$X \quad |b_n \sin(nx)||a_{p+n}| \le (t-A)(p-\delta)$$

So

$$\sum_{n=1}^{\infty} \Psi(n,\beta,\theta,p)\left[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)\right]b_n \sin(nx)\left|a_{p+n}\right|$$

since $b_n \le 1$ and for $-2\pi \le nx \le 2\pi$, we get $\sin(nx) \le 1$,then

$$\sum_{n=1}^{\infty} \Psi(n,\beta,\theta,p)\left[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)\right]b_n \sin(nx)\left|a_{p+n}\right|$$

$$\le \sum_{n=1}^{\infty} \Psi(n,\beta,\theta,p)\left[n(\lambda+\mu)+t(n+p-\delta)-A(p-\delta)\right]\left|a_{p+n}\right| \le (t-A)(p-\delta)$$

Then $Y(z,x) \in S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$.
So the proof is complete.
Let the function $f(x)$ defined by the form :

$$f(x) = \begin{cases} -1 & -\pi < x < 0 \\ 1 & 0 < x < \pi \end{cases}$$

The Fourier series of the function $f(x)$ is defined by the form :

$$R(x) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin(nx) .$$

Suppose that function $Q(z,x)$ be defined by the form:

$$Q(z,x) = f(z) * z^p (1 - z^n R(x))$$

$$= z^p + \sum_{n=1}^{\infty} \left|a_{p+n}\right| E(n) z^{p+n} , \quad (16)$$

where $E(n) = \frac{2(1-(-1)^n)}{n\pi} \sin(nx)$, (17)

and $f(z)$ defined by (2) .
In the next theorem , we show the function $Q(z,x)$ be in the class $S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$.

Theorem 5: Let $f(z) \in S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$ be defined by (2). Then the function $Q(z,x)$ defined by (16) be in the $S_p^*(A,\delta,\mu,\lambda,t,\beta,\theta)$, if $E(n) \le 1$, $-2\pi \le nx \le 2\pi$.

The proof of Theorem 5 is similar to proof of Theorem 4 .

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