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On Some forms of $M-$ continuous multifunctions

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المستخلص

المفتوحة المجموعة مفهوم بتقديم قمنا البحث هذا في $\theta m_X -$ بإعطاء قمنا المفهوم هذا على وبالاعتماد المستمرة القيم المتعددة بالدوال الخاصة المفاهيم بعض بتقديم قمنا كذالك .أخرى مفاهيم $M -$ بعض مع الدوال لهذه المكافات

Abstract

In this paper we introduce a new concept $-{}_X\theta m$ open set and some concepts are defined on it. Also, we introduce some concepts of$M-$ continuous multifunctions and we obtain some characterizations and some properties of multifunctions.

Key .words .and .phrases: $-{}_X m$ stricture, . $-{}_X m$ .continuous .multifunction, $-{}_X\theta m$ open set

INTRODUCTION

The concept of minimal structure space was introduced in 1996 by H.Maki in his work "On generalized semi –open and preopen sets "and the concept of multifunctions can be found in [1]. In 2000, Noiri and Popa [1] work on this space and introduce the concepts of lower\upper $M-$ continuous multifunction.

But in 1968 Velicko introduced the concept of θ −open set .This concept has since been studied intensively by many authors and they found that the collection of all θ −open sets in a topological space (X,Γ) forms a topology ${}_\theta\Gamma$ on X which is weaker than Γ , in this paper we join among those concepts and introduce some concept of $-{}_X\theta m$ open set. If we take X any non-empty set with ${}_X m$ is an m -structure s.t. ${}_X m$ satisfies the property (say γ): the intersection of a finite number of ${}_X m$ -open sets is ${}_X m$ -open , then the collection of all $-{}_X\theta m$ open sets forms a topology on X induced by ${}_X m$ and is denoted by ${}_{\theta m_X}\Gamma$, and as definitions of θ −derived (-border, -frontier, -exterior) sets in[2].We introduce $-{}_X\theta m$ derived(-

border, -frontier, -exterior) sets and finally we introduce some concepts of M_X -continuous multifunction , and we obtain some characterizations of such multifunction.

PRELIMINARIES

Definition.2.1 [1].A subfamily m_X of the power set $P(X)$ of a noneempty set X is called a minimal structure (briefly, m -structure) on X if $\varphi \in m_X$ and $X \in m_X$. Each member of m_X is said to be m_X -open and the complement of an m_X -open set is said to be m_X -closed set. We denote by (X,m_X) the m -structure space.

Definition.2.2 [1].Let (X,m_X) be an m -structure space, for a subset A of X , the m_X -closure of A and the m_X -interior of A are defined as follows:

(i) $m_X\text{-}cl(A) = \bigcap \{F : A \subseteq F, X \setminus F \in m_X\}$

(ii) $m_X\text{-}\mathrm{int}(A) = \bigcup \{U : U \subseteq A, U \in m_X\}$

Note that m_X - $cl(A)$ is not necessarily m_X -closed, also m_X - $\mathrm{int}(A)$ is not necessarily m_X -open.

Lemma.2.3 [1].Let (X,m_X) be an m -structure space , for a subset A of X , the following hold :

i- $m_X - cl(X \setminus A) = X \setminus m_X - \mathrm{int}(A)$ **and** $m_X - \mathrm{int}(X \setminus A) = X \setminus m_X - cl(A)$

ii- If $X \setminus A \in m_X$,then $m_X - cl(A) = A$ and if A $\in m_X$,then $m_X - \mathrm{int}(A) = A$

iii-If $A \subseteq B$, then $m_X - cl(A) \subseteq m_X - cl(B)$ and $m_X - \mathrm{int}(A) \subseteq m_X - \mathrm{int}(B)$

iv - $A \subseteq m_X - cl(A)$ and $m_X - \mathrm{int}(A) \subseteq A$

vi-$m_X - cl(m_X - cl(A)) = m_X - cl(A)$ and $m_X - \mathrm{int}(m_X - \mathrm{int}(A)) = m_X - \mathrm{int}(A)$

(1)A subset A of a topological space (X, τ) is called θ -open if

$A = \theta - \mathrm{int}(A) = \bigcup \{U : cl(U) \subseteq A, U \in \tau\}$

Lemma. 2.4[1]. Let (X,m_X) be an m -structure space and A a subset of X . Then $x \in m_X - cl(A)$ iff $U \cap A \neq \varphi$, for every $U \in m_X$ containing x .

Definition. 2.5[1]. An m -structure m_X on a non-empty set X is said to have the property (β) if the union of any family of subsets belonging to m_X belongs to m_X .

Lemma.2.6[1]. For an m -structure m_X on a non-empty set X , the following are equivalent:

i- m_X has property (β) .

ii- If $m_X - \mathrm{int}(V) = V$, then $V \in m_X$.

iii- If $m_X - cl(F) \quad F$ **, then** F **is** m *closed.* $_X -$

Lemma.2.7[1]. Let (X, m_X) **be an** m **-structure space with property** (β)**.For a subset** A **of** X **, the following properties hold:**

(i) $A \in m_X$ ***iff*** $m_X - \text{int}(A) \quad A$**.**

(ii) A **is** m *closed* $_X -$ ***iff*** $m_X - cl(A) \quad A$ **.**

(iii) $m_X - \text{int}(A) \in m_X$ **,and** $m_X - cl(A)$ **is** $-\ _X m$ **closed.**

$-\ _X \theta m$ **open**

Definition.3.1. Let (X, m_X) **be an** m **-structure space, for a subset** A **of** X **, the** $_X\theta m$ **-interior of** A **is defined by ;**

$_X\theta m$ **-int(** A **)=** $\{ : ( ) , \}\ _X\ _X \cup U\ m - cl\ U \subseteq A\ U \in m$ **.** A **is called** $_X\theta m$ **-open iff**

$m\ \text{int}(A)\ _X\theta = A$ **, and the complement of** A **is called** $_X\theta m$ **-closed.**

Remark. 3.2.Since every topology on X **is** m **-structure , then every** θ **-open is** $_X\theta m$ **-open .**

In general the concept of $_X m$ **-open sets and** $_X\theta m$ **-open sets are indepenent to see that we introduce the following example .**

Example.3.3. Let $X \quad \{1,2,3,4,5\}$ **and** $m\ \{X, \varphi, \{1,2\}, \{2,3\}, \{4\}, \{5\}\}\ _X$ **, then {1,2,3,5} is** $_X\theta m$ **-open but not** $_X m$ **-open and {2,3} is** $_X m$ **-open but not** $_X\theta m$ **-open.**

Remark.3.4. If an m **-structure space** $_X m$ **on a non-empty subset** X **satisfy** (β) **, then we have every** $_X\theta m$ **-open is** $_X m$ **-open.**

Remark.3.5. Let (X, m_X) **be an** m **-structure space .For subsets** A **and** B **of** X **. the following hold:**

(i) $m\ cl(A)\ X \setminus m\ \text{int}(A)\ _X\ _X \theta - \quad \theta -$ **and** $m\ \text{int}(X \setminus A)\ X \setminus m\ cl(A)\ _X\ _X \theta - \quad \theta -$ **.**

(ii) $m\ A\ m\ A\ A\ _X\ _X \theta - \text{int}( ) \subseteq - \text{int}( ) \subseteq$ **and** $A\ m\ cl(A)\ m\ cl(A)\ _X\ _X \subseteq - \subseteq \theta -$ **.**

(iii) If $A \subseteq B$ **, then** $m\ cl(A)\ m\ cl(B)\ _X\ _X \theta - \subseteq \theta -$ **and**

$m\ \text{int}(A)\ m\ \text{int}(B)\ _X\ _X \theta - \subseteq \theta -$ **.**

Proof: (i) $x\ m\ cl(X \setminus A)\ _X \in \theta - \Leftrightarrow U\ m\ x\ U\ _X \forall \in , \in$ **,**

$m - cl\ U\ X\ A \neq \varphi\ _X ( ) \cap \setminus \Leftrightarrow U\ m\ x\ U\ _X \forall \in , \in m\ cl\ U\ A\ _X - ( ) \not\subset \Leftrightarrow x\ m\ \text{int}(A)\ _X \notin \theta -$

$\Leftrightarrow x\ X \setminus m\ \text{int}(A)\ _X \in \theta -$ **.**

$x\ m\ \text{int}(X \setminus A)\ _X \in \theta - \Leftrightarrow\ _X \exists U \in m$ **s.t.** $x \in U$ **and**

$m\ cl\ U\ X\ A\ _X - ( ) \subseteq \setminus \Leftrightarrow\ _X \exists U \in m$ **s.t.** $x \in U$ **and** $m - cl\ U\ A \quad \varphi\ _X ( ) \cap$

$\Leftrightarrow x\ m\ cl(A)\ _X \notin \theta - - \Leftrightarrow x\ X \setminus m\ cl(A)\ _X \in \theta - -$ **.**

Remark.3.6. A **is** θm_x **-closed iff** $m_x\theta-cl(A) = A$ **.**

A **is** θm_x **-closed** $\Leftrightarrow X \setminus A$ **is** θm_x **-open** $\Leftrightarrow$

$\theta m_x\text{-int}(X \setminus A) = X \setminus A \Leftrightarrow X \setminus m_x\theta-cl(A) = X \setminus A \Leftrightarrow m_x\theta-cl(A) = A$ **.**

Remark .3.7. Let (X,m_x) **be an** m **-structure space and** A **,** B **are subsets of** X **, then :**

(i) $m_x\theta-cl(A \cup B) = m_x\theta-cl(A) \cup m_x\theta-cl(B)$ **.**

(ii) $m_x\theta-cl(A \cap B) \subseteq m_x\theta-cl(A) \cap m_x\theta-cl(B)$ **.**

(iii) $m_x\theta-\text{int}(A \cup B) \supseteq m_x\theta-\text{int}(A) \cup m_x\theta-\text{int}(B)$ **.**

(iv) $m_x\theta-\text{int}(A \cap B) \subseteq m_x\theta-\text{int}(A) \cap m_x\theta-\text{int}(B)$ **.**

Proof : (i) By remark (3.5,iii) we have

$m_x\theta-cl(A \cup B) \supseteq m_x\theta-cl(A) \cup m_x\theta-cl(B)$ **. Now let** $x \in m_x\theta-cl(A \cup B)$ **,**

$\forall U \in m_x$ **s.t.** $x \in U$ **, then** $m_x-cl(U) \cap (A \cup B) \neq \varphi \Rightarrow m_x-cl(U) \cap A \neq \varphi$

or $m_x-cl(U) \cap B \neq \varphi \Rightarrow x \in m_x\theta-cl(A) \cup m_x\theta-cl(B)$ **.**

(ii) By remark (3.5,iii).

(iii) Let $x \in m_x\theta-\text{int}(A) \cup m_x\theta-\text{int}(B) \Rightarrow x \in m_x\theta-\text{int}(A)$ **or**

$x \in m_x\theta-\text{int}(B)$ **.If** $x \in m_x\theta-\text{int}(A)$ **, then there exists** $U \in m_x$ **s.t.** $x \in U$ **,**

$m_x-cl(U) \subseteq A \subseteq A \cup B \Rightarrow x \in m_x\theta-\text{int}(A \cup B)$ **,and the same if**

$x \in m_x\theta-\text{int}(B)$

(iv) By remark (3.5,iii).

The equality of(ii,iii) and(iv) in remark (3.7) is not true in general .

Example.3.8.

(1) Let $X = \{1,2,3,4,5\}$ **,** $m_x = \{X, \varphi, \{1,2\}, \{3\}, \{1,2,3\}\}$ **and let** $A = \{1,2,3\}$**,**

$B = \{2,3,5\}$ **, then** $m_x\theta-cl(A) = X$ **and** $m_x\theta-cl(B) = X$ **, but**

$A \cap B = \{3\}$**,** $m_x\theta-cl(A \cap B) = \{3,4,5\}$ **, then**

$m_x\theta-cl(A \cap B) \neq m_x\theta-cl(A) \cap m_x\theta-cl(B)$ **.**

(2) Let $X = \{1,2,3,4,5\}$ **,** $m_x = \{X, \varphi, \{1,2\}, \{2,3,4\}, \{1,2,3\}, \{3,4,5\}, \{5\}\}$ **,**

let $A = \{1,2,3\}$**,** $B = \{4,5\}$ **, then** $m_x\theta-\text{int}(A) = \{1,2\}$ **,**

$m_x\theta-\text{int}(B) = \{5\}$ **and**

$A \cup B = X \Rightarrow m_x\theta-\text{int}(A \cup B) = X \Rightarrow$

$m_x\theta-\text{int}(A \cup B) \neq m_x\theta-\text{int}(A) \cup m_x\theta-\text{int}(B)$ **.**

Let A $\{1,2,3,4\}$ **and** B $\{3,4,5\}$ **, then** $m\ A\ A\ {}_x\theta - \mathrm{int}(\ )$ **and**

$m\ B\ B\ {}_x\theta - \mathrm{int}(\ )$ **.** $A\mathrm{I}\ B$ $\{3,4\}$ **,** $\theta - \mathrm{int}(\{3,4\})$ $\varphi\ {}_x m$ **,**

then $m\ \mathrm{int}(A\ B)\ m\ \mathrm{int}(A)\ m\ \mathrm{int}(B)\ {}_{x\,x\,x}\theta - \mathrm{I} \neq \theta - \mathrm{I}\theta -$ **.**

The ${}_x\theta m$ **-closure and** ${}_x\theta m$ **-interior of any nonempty proper subset of** $(\ ,\ )\ {}_x X\ m$ **are not idempotent i.e.**

$m\ cl(\ m\ cl(A))\ m\ cl(A)\ {}_{x\,x\,x}\theta - \theta - \neq \theta -$ **and**

$m\ \mathrm{int}(\ m\ \mathrm{int}(A))\ m\ \mathrm{int}(A)\ {}_{x\,x\,x}\theta - \theta - \neq \theta -$

We show that by the following example.

Example.3.9. X $\{1,2,3,4,5\}$ **and** $m\ \{X\ ,\varphi\ ,\{1\},\{1,2\},\{4,5\},\{2,3,5\}\}\ {}_x$ **and**

let A $\{1,2,3\}$**, then** $m - \mathrm{int}(A)$ $\{1,2\}\ {}_x\theta$ **,** $m - \mathrm{int}(\ m - \mathrm{int}(A))$ $\{1\}\ {}_{x\,x}\theta\ \theta$ **,**

thus $m\ \mathrm{int}(\ m\ \mathrm{int}(A))\ m\ \mathrm{int}(A)\ {}_{x\,x\,x}\theta - \theta - \neq \theta -$ **.If**

$m\ \{X\ ,\varphi\ ,\{1,2\},\{3\},\{1,2,3\}\}\ {}_x{}'$ **and**

A $\{3\}$**,** $m - cl(A)$ $\{3,4,5\}\ {}_x\theta \Rightarrow m\ cl\ m\ cl\ A\ X\ {}_{x\,x}\theta - (\theta - (\ )$ **, thus**

$m\ cl(\ m\ cl(A))\ m\ cl(A)\ {}_{x\,x\,x}\theta - \theta - \neq \theta -$ **.**

Note :1- The collection of all ${}_x\theta m$ **-open is denoted by** ${}_x\theta m$ **.**

2-An ***m*** **-structure** ***mX*** **on a non-empty set** ***X*** **is said to have property (**γ **) if the intersection of any finite number of subsets belonging to*****mX*** **belongs to*****mX*** **.**

3- If X **is a non-empty set and** ${}_x m$ **is an** m **-structure satisfies** γ **, then the following theorem gives a topology** ${}_{\theta m x}\Gamma$ **(consist of the collection of all** ${}_x\theta m$ **-open sets) induced by** ${}_x m$ **.**

Theorem.3.10. Let ***(X,mX)*** **be an** ***m*** **-structure space satisfies** γ **, then :**

(1) X,φ **are** ${}_x\theta m$ **-open.**

(2) the intersection of finite ${}_x\theta m$ **-open sets is** ${}_x\theta m$ **-open.**

(3) The union of any family of ${}_x\theta m$ **-open is** ${}_x\theta m$ **-open .**

Proof : (1) By remark (3.5(iv)).

2-Let A,B **be** ${}_x\theta m$ **-open sets , to prove** $A\mathrm{I}\ B$ **is** ${}_x\theta m$ **-open set i.e. to prove** $m\ A\ B\ A\ B\ \theta\ {}_x - \mathrm{int}(\ \mathrm{I}\ )$ I **.** $m\ A\ B\ A\ B\ \theta\ {}_x - \mathrm{int}(\ \mathrm{I}\ ) \subseteq \mathrm{I}$ **(remark 3.5, ii) ,**

now let $x \in A\mathrm{I}\ B \Rightarrow x \in A$ **and** $x \in B \Rightarrow x\ m\ \mathrm{int}(A)\ {}_x \in \theta -$ **and**

$x\ m\ \mathrm{int}(B)\ {}_x \in \theta -$ **, then** $U\ m\ x\ U\ {}_x \exists \in , \in$ **s.t.** $m\ cl\ U\ A\ {}_x - (\ ) \subseteq$ **and**

$V\ m\ x\ V\ x \exists \in , \in$ **s.t.** $m\ cl\ V\ B\ x - ( ) \subseteq$ **,**

then$m\ cl\ U\ V\ m\ cl\ U\ m\ cl\ V\ A\ B\ x - (\mathrm{I}) \subseteq x - ( ) \mathrm{I}\ x - ( ) \subseteq \mathrm{I}$ **, since** $x\ U\ \mathrm{I} V \in m$

and $x \in U\ \mathrm{I} V$ **, then** $x\ m\ \mathrm{int}(A\ B) \in \theta\ x - \mathrm{I}$ **, that is** $m\ A\ B\ A\ B\ \theta\ x - \mathrm{int}(\mathrm{I})\ \ \mathrm{I}$ **.**

3- Let $\in {}_i A\ x\ \theta m$ **,** $\forall i \in I$ **, to prove** $\in {}_i \cup A\ x\ \theta m$ **i.e. to prove**

$x\ i\ i A\ A\ m \cup \cup\ \ -) \mathrm{int}(\theta$ **. Now , let** $\bigcup$

$I\ i$

$i\ x\ A$

$\in$

$\in \Rightarrow {}_i x \in A$ **, for some** i **then**

$\mathrm{int}( )\ x\ I\ x \in m - A$ **therefore there is**

$U\ m\ x\ U\ x \in \in ,$ **s.t** $\bigcup$

$I\ i$

$x\ i\ i\ m\ cl\ U\ A\ A$

$\in$

$\subseteq \subseteq -) ($ **therefore** $) \mathrm{int}(\bigcup$

$I\ i$

$x\ i\ x\ m\ A$

$\in$

$\in \theta -$ **then**

by remark(3.5(ii)) we have $\bigcup\ \bigcup$

$i\ I$

i

$i\ I$

$x\ i\ m\ A\ A$

$\in \in$

$\theta - \mathrm{int}( )$ **.**

Remark.3.11. If X **is a non empty set and** $x\ m$ **is a** $m -$ **structure which does not satisfy (**γ **) property then the intersection of finite** $- x\ \theta m$ **open may not be** $- x\ \theta m$ **open; we show that by the following example.**

Example.3.12.Let ($x\ X\ ,m$ **)be an** $m -$ **structure space as in example (3.8(2)) .Let** A **{1,2,3,4},** B **={3,4,5},then** $A, B\ x \in \theta m$ **,but** $x\ A \mathrm{I}\ B \notin \theta m$ **.**

Remark.3.13. .Let ($x\ X\ ,m$ **) be an** $m -$ **structure space and** $A \subseteq X$ **, then**

i-Let $x\ U \in m$ **and** $A \subseteq X$ **,if**$U\ \mathrm{I}\ A\ \ \varphi$ **, then**$m - cl\ A\ U\ \ \varphi\ x ( ) \mathrm{I}$ **.**

ii- $m\ cl(A)\ x\ \theta -$ **is** $x\ m$ **-closed set.**

iii-If $x\ U \in m$ **,then**$m\ cl(U)\ m\ cl(U)\ x\ x -\ \ \theta -$ **.**

iv-If $_x U \in m$ **and** U **is an** $_x m$ **-closed set, then** U **is** $- {_x}\theta m$ **closed.**

v- If $_x U \in m$ **and** $_x m$ **satisfies (**β **) property, then** U **is** $_x m$ **-closed set iff** U **is**

$- {_x}\theta m$ **closed.**

The converse of (iv) in the remark (3.13) is not true in general as the following example shows

Example.3.15.Let $X \quad \{1,2,3,4\}$**,** $m\ \{X, \varphi, \{2,3\},\{1,5\},\{1,4\}\}\ _x$ **and**

let $A \quad \{2,3\}$**, then** $_x A \in m$ **and** A **is** $- {_x}\theta m$ **closed but not** $_x m$ **-closed.**

Some properties of $- {_x}\theta m$ open sets

The following definitions and results are the $_x\theta m$ **version of those definitions and results in [2].**

Definition.4.1.Let ($_x X, m$ **) be a** $m -$ **structure space and** $A \subseteq X$ **.A point** $x \in X$ **is said to be** $- {_x}\theta m$ **limit point of** A **if for each**

$U \in {_x}\theta m$ **s.t.** $x \in U$**, then** $U \cap (A \setminus \{x\}) \neq \varphi$ **. The set of all** $- {_x}\theta m$ **limit points of** A **is called** $- {_x}\theta m$ **derived set of** A **and is denoted by**

$m\ d(A)\ _x\theta -$ **.**

Theorem.4.2.Let ($_x X, m$ **) be an** $m -$ **structure space and**

$A, B \subseteq X$ **,then the following hold:-**

i-If $A \subseteq B$**,then** $m\ d(A)\ m\ d(B).\ _x\ _x\theta - \subseteq \theta -$

ii- $m\ d(A\ B)\ m\ d(A)\ m\ d(B).\ _x\ _x\ _x\theta - \cap \subseteq \theta - \cap \theta -$

iii- $m\ d(A)\ m\ d(B)\ m\ d(A\ B).\ \theta\ _x - \cup \theta\ _x - \subseteq \theta\ _x - \cup$

iv- $m - d\ m - d\ A\ A \subseteq\ _x\ _x\theta\ (\theta\ (\ ))\setminus m\ d(A).\ _x\theta -$

v- $m\ d\ A \cup m\ d\ A\ A \cup\ _x\ _x\theta - (\ \theta - (\ )) \subseteq m\ d(A).\ _x\theta -$

proof:-(i)by definition .(ii)and (iii)by (i)

(iv) Let $x\ m\ d\ m\ d\ A\ A\ _x\ _x \in \theta - (\theta - (\ ))\setminus$ **,and** $_x U \in m$ **s.t.** $x \in U$ **,then**

$U \cap \theta m - d(A)\setminus\{x\} \neq \varphi\ _x$ **,let** $y \in U \cap m\ d(A)\setminus\{x\}\ _x\theta -$ **,therefore** $y \in U$ **and**

$y\ m\ d(A)\ _x \in \theta -$ **,then** $U \cap A \setminus \{y\} \neq \varphi$ **,let** $z \in U \cap A \setminus \{y\}$**,then** $z \in U$ **and**

$z \in A \setminus \{y\}$

$z \neq x$ **, hence** $z \in U \cap A \setminus \{x\}$**and**$U \cap A \setminus \{x\} \neq \varphi$ **, then** $x\ m\ d(A)\ _x \in \theta -$

(v)Let $x\ m\ d(A\ m\ d(A))\ _x\ _x \in \theta - \cup \theta -$ **,if** $x \in A$**,then** $x\ A\ m\ d(A)\ _x \in \cup \theta -$ **and if**

$x \notin A$**,then** $x\ m\ d\ A\ m\ d\ A\ A\ _x\ _x \in \theta - (\ \cup \theta - (\ ))\setminus$ **, hence for any** $U\ m\ x\ U\ _x \in \theta\ , \in$

$U\{A\ \theta m - d(A)\}\setminus\{x\} \neq \varphi\ \text{I}\ \cup\ _x$ **, then** $U\ \text{I}\ A\setminus\{x\} \neq \varphi$ **or**

$U\ \theta m - d(A)\setminus\{x\} \neq \varphi\ \text{I}\ _x$

, then by (iv) $x\ m\ d(A)\ _x \in \theta -$ **.**

Definition .4.3.Let ($_x X, m$ **) be a** $m -$ **structure space and** $A \subseteq X$ **.the set** $m\ b(A)\ A\setminus m\ \text{int}(A)\ _x\ _x\ \theta - \quad \theta -$ **is said to be** $- _x\theta m$ **border of** A **.**

Theorem.4.4. Let ($_x X, m$ **) be a** $m -$ **structure space and** $A \subseteq X$ **,then the following hold:-**

i- $A\ m\ \text{int}(A)\ m\ b(A).\ _x\ _x\quad \theta - \cup \theta -$

ii-$\theta m - \text{int}(A)\ \theta m - b(A) \quad \varphi.\ _x\ \text{I}\ _x$

iii- A **is** $- _x\theta m$ **open iff**$\theta m - b(A) \quad \varphi\ _x$ **.**

iv-$\theta m - \text{int}(\theta m - b(A)) \quad \varphi.\ _x\ _x$

v- $m\ b(A)\ A\ m\ cl(A).\ _x\ _x\ \theta - \quad \text{I}\theta -$

Proof :-(i),(ii)and(iii)are trvial.(iv) Suppose $\theta m - \text{int}(\theta m - b(A)) \neq \varphi\ _x\ _x$ **then**$\exists x \in m\ \text{int}(\ m\ b(A))\ _x\ _x\ \theta - \theta -$ **,**

hence $x \in m\ b(A)\ _x\ \theta -$ **.Now since** $m - \text{int}(\ m - b(A)) \subseteq\ _x\ _x\ \theta\ \theta\ m\ \text{int}(A)\ _x\ \theta -$ **,then** $x \in m\ b(A)\ _x\ \theta - m\ \text{int}(A)\ _x\ \text{I}\theta -$ **that is contradiction, therefore**

$\theta m - \text{int}(\theta m - b(A)) \quad \varphi.\ _x\ _x$

(v) $m\ b(A)\ A\setminus m\ \text{int}(A)\ _x\ _x\ \theta - \quad \theta - = A\setminus\{X\setminus m\ cl(X\setminus A)\}\ _x\ \theta - =$

$A\ m\ cl(X\setminus A)\ _x\ \text{I}\theta -$ **.**

Definition .4.5. Let ($_x X, m$ **) be an** $m -$ **structure space and** $A \subseteq X$ **.The set**

$m\ fr(A)\ m\ cl(A)\setminus m\ \text{int}(A)\ _x\ _x\ _x\ \theta - \quad \theta - \theta -$ **is said to be** $- _x\theta m$ **frontier of** A **.**

Theorem.4.6. Let ($_x X, m$ **) be a** $m -$ **structure space and** $A \subseteq X$ **,then the following hold:-**

i- $m - cl(A) \quad _x\ \theta\ m\ fr(A)\ m\ \text{int}(A)\ _x\ _x\ \theta - \cup \theta -$ **.**

ii-$\theta m - fr(A)\ \theta m - \text{int}(A) \quad \varphi\ _x\ \text{I}\ _x$ **.**

iii- $m\ b(A)\ m\ fr(A)\ _x\ _x\ \theta - \subseteq \theta -$ **.**

iv- $m\ fr(A)\ m\ cl(A)\ m\ cl(X\setminus A)\ _x\ _x\ _x\ \theta - \quad \theta - \text{I}\theta -$ **.**

v- $m\ fr(A)\ m\ fr(X\setminus A)\ _x\ _x\ \theta - \quad \theta -$ **.**

vi- $m\ \text{int}(A)\ A\setminus m\ fr(A)\ _x\ _x\ \theta - \quad \theta -$ **.**

Proof :-(i),(ii)and(iii)by definition.

(iv) $m - cl(A)\ m - cl(X\setminus A) \quad _x\ _x\ \theta\ \text{I}\theta\ m\ cl(A)\ (X\setminus m\ \text{int}(A))\ _x\ _x\ \theta - \text{I}\ \theta - =$

$m\,cl(A) \setminus m\,\text{int}(A)\ {}_{x}{}_{x}\theta - \theta - = m\,fr(A)\ {}_{x}\theta -$.

(v) $m\,fr(X \setminus A)\ {}_{x}\theta - = m\,cl(X \setminus A) \setminus m\,\text{int}(X \setminus A)\ {}_{x}{}_{x}\theta - \theta - =$

$X \setminus m\,\text{int}(A) \setminus X \setminus m\,cl(A)\ {}_{x}{}_{x}\theta - \theta - = m\,cl(A) \setminus m\,\text{int}(A)\ {}_{x}{}_{x}\theta - \theta - = m\,fr(A)\ {}_{x}\theta -$.

The following example show that the frontier set in general is not $-\ {}_{x}\theta m$ closed

Example.4.7. Let ($_{x}X, m$) be an $m -$ structure space as in example (4.9), and let

$A \quad \{3\}$**, then** $m - cl(A) \quad \{3,4,5\}\ {}_{x}\theta$ **,and**$\theta m - \text{int}(A) \quad \varphi\ {}_{x}$**, then**

$m\,fr\,A\,A\ {}_{x}\theta - (\ )$ **,but** A **is not** $-\ {}_{x}\theta m$ **closed.**

Definition.4.8. Let ($_{x}X, m$) be an $m -$ structure space and $A \subseteq X$ .The set

$m\,ext(A)\ m\,\text{int}(X \setminus A)\ {}_{x}{}_{x}\theta - \quad \theta -$ **is said to be** $-\ {}_{x}\theta m$ **exterior of** A **.**

Theorem.4.9. Let ($_{x}X, m$) be a $m -$ structure space and $A \subseteq X$,then the following are hold:-

i- $-\ {}_{x}\theta m\,ext(A)$ **is** $-\ {}_{x}m$ **open set.**

ii- $m\,ext(\ m\,ext(A))\ m\,\text{int}(\ m\,cl(A))\ {}_{x}{}_{x}{}_{x}{}_{x}\theta - \theta - \quad \theta - \theta -$.

iii-If $A \subseteq B$**, then** $-\ {}_{x}\theta m\,ext(B) \subseteq -\ {}_{x}\theta m\,ext(A)$ **.**

iv- $m\,ext(A\ B)\ \theta\ {}_{x} - \cup \subseteq -\ {}_{x}\theta m\,ext(A) \cup -\ {}_{x}\theta m\,ext(B)$ **.**

v- $-\ {}_{x}\theta m\,ext(A)\ \text{I} -\ {}_{x}\theta m\,ext(B) \subseteq m\,ext(A\ B)\ \theta\ {}_{x} - \text{I}$ **.**

vi-$\theta m - ext(X\ ) \quad \varphi\ {}_{x}$ **and** $m\,ext\,X\ {}_{x}\theta - (\varphi\ )$ **.**

vii- $m\,ext(X \setminus m\,ext(A))\ m\,ext(A).\ {}_{x}{}_{x}{}_{x}\theta - \theta - \subseteq \theta -$

viii- $m\,\text{int}(A)\ m\,ext(\ m\,ext(A))\ {}_{x}{}_{x}{}_{x}\theta - \subseteq \theta - \theta -$.

ix- $X \quad m\,\text{int}(A)\ {}_{x}\theta - m\,ext(A)\ {}_{x}\cup\theta - m\,fr(A)\ {}_{x}\cup\theta -$.

Proof :-(i) by remark (3.13(2))

(ii) $m\,ext(\ m\,ext(A))\ {}_{x}{}_{x}\theta - \theta - = m\,\text{int}(X \setminus m\,ext(A))\ {}_{x}{}_{x}\theta - \theta - =$

$m\,\text{int}(X \setminus m\,\text{int}(X \setminus A))\ {}_{x}{}_{x}\theta - \theta - = m\,\text{int}(\ m\,cl(A))\ {}_{x}{}_{x}\theta - \theta -$

(iii)it is clear (iv)and(v)by(iii)

(vii) $m\,ext(X \setminus m\,ext(A))\ m\,ext(X \setminus m\,\text{int}(X \setminus A))\ {}_{x}{}_{x}{}_{x}{}_{x}\theta - \theta - \quad \theta - \theta - =$

$m\,\text{int}(X \setminus X \setminus m\,\text{int}(X \setminus A))\ {}_{x}{}_{x}\theta - \theta - = m\,\text{int}(\ m\,\text{int}(X \setminus A))\ {}_{x}{}_{x}\theta - \theta -$

$m\,\text{int}(X \setminus A)\ {}_{x}\subseteq\theta - = m\,ext(A)\ {}_{x}\theta -$.

(viii) $m - \text{int}(A) \subseteq {}_{x}\theta\ m\ \text{int}(\ m\ cl(A))\ {}_{x\,x}\theta - \theta - = m\ \text{int}(X \setminus m\ \text{int}(X \setminus A))\ {}_{x\,x}\theta - \theta - =$

$m\ \text{int}(X \setminus m\ ext(A))\ {}_{x\,x}\theta - \theta - = m\ ext(\ m\ ext(A)).\ {}_{x\,x}\theta - \theta -$

(ix) $m\ \text{int}(A)\ {}_{x}\theta - m\ ext(A)\ {}_{x}\cup\theta - m\ fr(A)\ {}_{x}\cup\theta - =$

$m\ \text{int}(A)\ {}_{x}\theta - \cup\ m\ \text{int}(X \setminus A)\ {}_{x}\theta - \cup\ m\ cl(A) \setminus m\ \text{int}(A)\ {}_{x\,x}\theta - \theta - =$

$m\ \text{int}(X \setminus A)\ {}_{x}\theta - m\ cl(A)\ {}_{x}\cup\theta - = X \setminus m\ cl(A)\ {}_{x}\theta - m\ cl(A)\ {}_{x}\cup\theta - = X$ **.**

Some forms of $M-$ continuous multifunction

Recall that a multifunction $F : X \to Y$ **from a topological space** (X,τ) **into a topological space** (Y,σ) **is a point to set correspondence with** $F(x) \neq \varphi$ **for all** $x \in X$ **.**

Definition.5.1. [1].Let $F : (X,\tau) \to (Y,\sigma)$ **be a multifunction from a topological space** (X,τ) **into a topological space** (Y,σ) **.The upper and lower inverse of a set** B **of the space** Y **are denoted by** $F\ (B)$ **and** $F^{-}(B)$ **, respectively are defined as**

$F\ (B) \quad \{x \in X : F(x) \subseteq B\}$ **,** $F^{-}(B) \quad \{x \in X : F(x) \cap B \neq \varphi\}$ **.**

And $X \setminus F^{-}(K) \quad F\ (Y \setminus K)$

Definition.5.2. [1] Let (${}_{x}X, m$ **) and** $(\ ,\ )\ {}_{Y}Y\ m$ **be a** $m-$ **structure spaces. A multifunction** $F : (\ ,\ )\ {}_{X}X\ m \to (\ ,\ )\ {}_{Y}Y\ m$ **is said to be**

1-upper$M-$ **continuous if for** $x \in X$ **and each** ${}_{Y}V \in m$ **containing** $F(x)$ **,there exists** ${}_{x}U \in m$ **containing** x **s.t.** $F(U) \quad \cup\{\ f(x) : x \in U\} \subseteq V$ **.**

2-Lower $M-$ **continuous if for** $x \in X$ **and each** ${}_{Y}V \in m$ **s.t.** $F(x) \cap V \neq \varphi$ **,** **there exists** ${}_{x}U \in m$ **containing** x **s.t.** $F(u) \cap V \neq \varphi$ **for each**$u \in U$ **.**

Definition.5.3. Let (${}_{x}X, m$ **) and** $(\ ,\ )\ {}_{Y}Y\ m$ **be** $m-$ **structure spaces. A multifunction** $F : (\ ,\ )\ {}_{X}X\ m \to (\ ,\ )\ {}_{Y}Y\ m$ **is said to be upper**$\theta M-$ **continuous (upper**$\theta^{*}M-$**continuous), if for** $x \in X$ **and each** ${}_{Y}V \in \theta m$ **containing** $F(x)$ **,there exists** ${}_{x}U \in m$ **containing** x

s.t. $F(U) \subseteq V$ **(** $F\ m\ cl\ U\ V\ {}_{x}(\ -\ (\ )) \subseteq$ **,receptivity)**

Definition.5.4. Let (${}_{x}X, m$ **) and** $(\ ,\ )\ {}_{Y}Y\ m$ **be** $m-$ **structure spaces. A multifunction** $F : (\ ,\ )\ {}_{X}X\ m \to (\ ,\ )\ {}_{Y}Y\ m$ **is said to be an upper strong** $\theta M-$ **continuous (briefly u.s.**θM **- continuous), if for** $x \in X$ **and each** ${}_{Y}V \in m$ **containing** $F(x)$ **,there exists** ${}_{x}U \in m$ **containing** x

s.t. $F^{-}(m_X\text{-}cl(U)) \subseteq V$.

Remark.5.5.Every u.s.θM-continuous multifunction is an upper$M-$ continuous, but the converse in general is not true to see that by the following example.

Example.5.6. Let $X = R$ **(real numbers) with the topology τ generated by ths basis with members of the form** (a,b) **and** $(a,b)\setminus K$ **,where** $K = \{\frac{1}{n} : n \in N\}$ **,let** $F:(X,\tau)\to(X,\tau)$ **be the identity map, then** F **is upper$M-$ continuous but not u.s.θM-continuous.**

Theorem.5.7.For a multifunction $F:(X,m_X)\to(Y,m_Y)$ **,the following properties are equivalent:-**

i- F **is u.s.θM-continuous.**

ii- $F^{+}(V) = m_X\text{-}\theta\text{-}int(F^{+}(V))$ **for every** $V \in m_Y$ **.**

iii- $F^{-}(K) = m_X\text{-}\theta\text{-}cl(F^{-}(K))$ **for every** m_Y- **closed set** K **.**

proof :-(i)→(ii)Let $x \in F^{+}(V)$ **,then** $F(x) \subseteq V$ **,by(i) there exists** $U \in m_X, x \in U$ **s.t.** $F(m_X\text{-}cl(U)) \subseteq V$ **,thus** $m_X\text{-}cl(U) \subseteq F^{+}(V)$ **hence** $x \in m_X\text{-}\theta\text{-}int(F^{+}(V))$ **,that is** $F^{+}(V) \subseteq m_X\text{-}\theta\text{-}int(F^{+}(V))$ **,and by definition** $F^{+}(V) = m_X\text{-}\theta\text{-}int(F^{+}(V))$ **.**

(ii) →(i) let $x \in X$ **and** $V \in m_Y$ **s.t.** $F(x) \subseteq V$ **,then** $x \in F^{+}(V)$ **,that is** $x \in m_X\text{-}\theta\text{-}int(F^{+}(V))$ **,hence there exits** $U \in m_X$ **s.t.** $x \in U$ **and** $m_X\text{-}cl(U) \subseteq F^{+}(V)$

That is $F(m_X\text{-}clU) \subseteq V$ **, then** F **is u.s.θM-continuous .**

(ii) → (iii) Let K **be any** m_Y- **closed set, since** $Y\setminus K \in m_Y$ **,and**

$X \setminus F^{-}(K) \quad F \quad (Y \setminus K)$ **then** $X \setminus F^{-}(K) = m\ \mathrm{int}(F(Y \setminus K))_{X}$

$\theta - \ =$

$m\ \mathrm{int}(X \setminus F(K))_{X}$

$\theta - {}_{-} = X \setminus m\ cl(F(K))_{X}$

$\theta - {}_{-}$ **that is** $F^{-}(K) = m\ cl(F(K))_{X}$

$\theta - {}_{-}$.

(iii)→ (ii)Let ${}_{Y}V \in m$ **,then** $X \setminus F \quad (V) = F^{-}(Y \setminus V) = m\ cl(F(Y \setminus V))_{X}$

$\theta - {}_{-} =$

$m\ cl(X \setminus F(V))_{X}$

$\theta - \ = X \setminus m\ \mathrm{int}(F(V))_{X}$

$\theta -$ **,then** $F \quad (V) = m\ \mathrm{int}(F(V))_{X}$

$\theta -$.

Theorem.5.8.For a u.s.θM **-continuous multifunction**

$F : (,)_{X} X\ m \to (,)_{Y} Y\ m$ **,where** $(,)_{Y} Y\ m$ **satisfies** (β)**property , the following hold :**

(i) $m - cl(F^{-}(B)) \subseteq {}_{X}\theta\ F(\ m\ cl(B))_{Y^{-}}\theta -$ **for every subset** B **of** Y **.**

(ii) $F \quad (\ m - \mathrm{int}(B)) \subseteq {}_{Y}\theta\ m\ \mathrm{int}(F(B))_{X}$

$\theta -$ **for every subset** B **of** Y **.**

proof :(i)Let B **be any subset of** Y **by lemma (2.7(ii))** $m\ cl(B)_{Y} -$ **is**

$-{}_{Y}m$ **closed ,then** $F(B)\ F(m\ cl(B))_{Y^{-}} \subseteq {}_{-} - = m\ cl(F(m\ cl(B))_{X\,Y}\theta - {}_{-} -$ **(by theorem(5.8)).Now ,let**

$x \in m\ cl(F(B))_{X}$

$\theta - {}_{-}$ **,then for any** $U\ m\ x\ U_{X} \in , \in$ **,**$m\ cl(U)\ F^{-}(B) \neq \varphi_{X}$ I **,then**

$m\ cl(U)\ F^{-}(m - cl(B)) \neq \varphi_{X}$ I ${}_{Y}$**,hence** $x \in m\ cl(F(m\ cl(B))_{X\,Y}\theta - {}_{-} - =$

$F(m\ cl(B)_{Y^{-}} -$ **,then** $x\ F(m\ (cl(B))\ F(\ m\ cl(B))_{Y\,Y} \in {}_{-} - \subseteq {}_{-}\theta -$ **therefore**

$m - cl(F^{-}(B)) \subseteq {}_{X}\theta\ F(\ m\ cl(B))_{Y^{-}}\theta -$ **.**

(ii) Let B **any set of** Y **, then** $X \setminus m - \mathrm{int}(F \quad (B)) \quad m - cl(X \setminus F \quad (B)) \quad {}_{X\,X}\theta\ \theta$

$m\ cl(F(Y \setminus B))_{X}$

$\theta - {}_{-}$

$by(i)$

$\subseteq F(\ m\ cl(Y \setminus B))_{Y^{-}}\theta - = F(Y \setminus m\ \mathrm{int}(B))_{Y^{-}}\theta - =$

$X \setminus F(\ m\ \mathrm{int}(B))_{Y} \quad \theta -$ **then** $F \quad (\ m - \mathrm{int}(B)) \subseteq {}_{Y}\theta\ m\ \mathrm{int}(F(B))_{X}$

$\theta -$.

Remark.5.9.The converse of theorem (5.8) is not true in general we show that in the following example

Example.5.10.Let $X \{1,2,3\}$ **and** $Y \{a,b,c\}$ **with** $m \{X, \varphi, \{1\}\}_X$ **and** $m \{Y, ,\{a\},\{a,b\},\{a,c\}\}_Y \varphi$ **, define** $F : X \to Y$ **by** $F(1) \{a\}$, $F(2) \{c\}$ **and** $F(3) \{a,b\}$**,then** ${}_Y m$ **satisfies** (β) **property and**

$F (\theta m - \text{int}(A)) \varphi\ {}_Y \subseteq m\ \text{int}(F(A))_Y$

$\theta -$ **for every** $A \subset Y$ **and**

$F\ m\ X\ X_Y\ (\theta - \text{int}(\ )) = m\ \text{int}(F(X))_Y$

$\theta -$ **then for any subset** A **of** Y

$F(m\ \text{int}(A))_Y\ \theta - \subseteq m\ \text{int}(F(A))_Y$

$\theta -$ **and any non empty subset** A **of** Y

we have

$m\ cl(A)_Y \theta - = Y$ **,then** $F(m\ cl(A))_Y -\theta - = F^-(Y)\ X$ **,then**

$m\ cl\ F\ A\ X_Y \theta - (\ -(\ )) \subseteq =$

$F(m\ cl(A))_Y -\theta -$ **,and** $\theta m - cl(F^-(\varphi))\ \varphi\ {}_Y F(\theta m\ cl(\varphi))_Y - -$ **,but** F **is not u.s.**θM **-continuous, since** $1 \in X$ **and** ${}_Y F(1) \{a\} \subseteq \{a\} \in m$ **,but the only** ${}_X m$ **-0pen contain 1are** $X, \{1\}$ **and** $F(m\ cl(\{1\})_X -$ $= F(m - cl(\{X\})\ {}_X F(X)\ Y \not\subset \{a\}$**.**

Remark.5.11.(i)Every upper$\theta_* M$ **−continuous is upper**θM **− continuous but the converse in general is not true and the concept of upper**M **− continuous and upper**$\theta_* M$ **− continuous are disjoint as the following examples show**

Examples.5.12.(i) Let $X \{1,2,3,4\}, m \{\ , X, \{1\},\{3\},\{4\},\{3,4\}\}_X\ \varphi$ **and** $Y \{a,b,c,d\}$**,**$m \{\ , Y, \{a\},\{a,b\},\{b,d\},\{c,d\},\{a,c,d\}\}_Y\ \varphi$ **,define** $F : X \to Y$ **by** $F(1) \{a\}$ **,** $F(2) \{a,b,c\}$**,**$F(3) \{b\}\ F(4) \{d\}$**, then it is clear that** F **is upper**θM **− continuous but not upper**$\theta_* M$ **−continuous, since the set of all −** ${}_Y \theta m$ **open is** $\{\varphi, Y, \{a\},\{a,b\},\{c,d\},\{b,c,d\},\{a,c,d\}\}$ **and** $(1)(\{\ \})\{\ \}(\ )_Y F\ a \subseteq a \in m$ **.**

But the only − ${}_X \theta m$ **open set contain 1 are {1},** X **and** $F(m\ cl(\{1\})\ F(\{1,2\})\ \{a,b,c\}\ \{a\}_X - \not\subset$ **.**

(ii)In example (5.10) F **is upper**θM **− continuous since the only**

$-\ _Y\theta m$ **open sets are** Y,φ **and for each** $x\in X$, $F(x)\subseteq Y$ **there exists**
$_xX\in m$ **,s.t.**
$F(X)\subseteq Y$**, but** F **is not upper**$M-$ **continuous since**
$2\in X$ **and** $F(2)\ \ \{c\}\subseteq\{a,c\}$**,**
then $_x\forall U\in m$ **s.t.** $2\in U$ **,** $F(U)\not\subset\{a,b\}$**.**
(iii)Let $X\ \ \{1,2,3\}$**and**$Y\ \ \{a,b,c,d\}$**with**$m\ \{X,\varphi,\{1\}\}\ _x$ **and**
$m\ \{Y,\ ,\{a\},\{a,b\},\{b,d\},\{c,d\},\{a,c,d\}\}\ _Y\ \ \varphi$ **, define** $F:X\to Y$ **by** $F(1)\ \ \{a\}$
$F(2)\ \ \{b,c\},F(3)\ \ Y$**, then** F **upper**$M-$ **continuous since** $1\in X$ **and**
$F(1)\ \ \{a\}$
and $_Y\{a\}\subseteq\{a\}(\{a,c,d\},X)\in m$ **,then there exists** $_x\{1\}\in m$ **s.t.** $F(\{1\})\subseteq\{a\}$ **,**
$2\in X$ **and** $_YF(2)\ \ \{b,c\}\subseteq Y\in m$ **,there is** $_xX\in m$ **s.t.** $F(X)\subseteq Y$ **,and** $3\in X$
similar,
but F **is not upper**$\theta M-$ **continuous ,since the set of all** $-\ _Y\theta m$ **open is**
$\{\varphi,Y,\{a\},\{a,b\},\{c,d\},\{b,c,d\},\{a,c,d\}\}$**then** $2\in X$ **and**
$F(2)\ \ \{b,c\}\subseteq\{b,c,d\}$**and for every** $_xU\in m$ **s.t.** $2\in U$ $F(U)\not\subset\{b,c,d\}$ **.**

Remark.5.13. Let ($_xX,m$ **) and** $(\ ,\ )\ _YY\ m$ **be** $m-$ **structure spaces**
s.t $(\ ,\ )\ _YY\ m$ **satisfies**
(β)**property ,then every upper** $M-$ **continuous is**
upper$\theta M-$ **continuous.**

Theorem.5.14.For a multifunction $F:(\ ,\ )\ _xX\ m\to(\ ,\ )\ _YY\ m$ **,the following properties are equivalent:-**

i- F **is upper**$\theta\ _*M-$**continuous.**
ii- $F\ \ (V)=m\ \mathrm{int}(F\ (V))\ _x$
$\theta-$ **for every** $_YV\in\theta m$ **.**
iii- $F^-(K)=m\ cl(F\ (K))\ _x$
$\theta-$ $_-$**for every** $-\ _Y\theta m$ **closed set** K **.**
iv- $m-cl(F^-(B))\subseteq\ _x\theta\ F\ (m\ cl(B))\ _{Y}{}_-\theta-$ **for every subset** $B\in\ _Ym$ **.**
v- $F\ \ (m-\mathrm{int}(B))\subseteq\ _Y\theta\ m\ \mathrm{int}(F\ (B))\ _x$
$\theta-$ **for every subset** $B\in\ _Ym$ **.**

Proof :-(i)→ (ii) → (iii) → (iv) → (v) (similar to the proof of theorem (5.7)and (5.8)).

(v) →(i)Let $x\in X$ **and** $V\in \theta m_Y$ **s.t.** $F(x)\in V$ **,then** $x\in F^{+}(V)$ **,therefore**

$x\in F^{+}(m_Y-\text{int}(V))\subseteq \theta m_X\text{int}(F^{+}(V))$

, then there exists $U\in m_X$ **s.t.**

$x\in U$

and $m_X cl(U)\subseteq F^{+}(V)$ **,that is** $F(m_X-cl(U))\subseteq V$ **,then** F **is upper**

$\theta_* M$ **−continuous.**

Definition.5.15. Let (X, m_X **) and** (Y, m_Y) **be** $ms-$ **structure spaces. A multifunction** $F:(X,m_X)\to(Y,m_Y)$ **is said to be lower** $\theta M-$ **continuous (lower** $\theta_* M$ **−continuous) if for** $x\in X$ **and each** $V\in \theta m_Y$ **s.t** $F(x)\cap V\neq\varphi$ **,there exists** $U\in m_X$ **containing** x **s.t.** $F((u))\cap V\neq\varphi$ **(** $F(m_X-cl(u))\cap V\neq\varphi$ **,respectively)** $\forall u\in U$ **.**

Definition.5.16. Let (X, m_X **) and** (Y, m_Y) **be** $m-$ **structure spaces. A multifunction** $F:(X,m_X)\to(Y,m_Y)$ **is said to be lower strong** $\theta M-$ **continuous (briefly 1.s.**θM **- continuous), if for** $x\in X$ **and each** $V\in m_Y$ **s.t** $F(x)\cap V\neq\varphi$ **,there exists** $U\in m_X$ **containing** x **s.t.** $F(m_X-cl(u))\cap V\neq\varphi$ $\forall u\in U$ **.**

Remark.5.17. Every lower $M-$ **continuous is l.s.**θM **-continuous , but the converse is not true in general .as the following example shows .**

Example.5.18. Let $X=\{1,2,3\}$ **and** $Y=\{a,b,c,d\}$ **s.t.**

$m_X=\{X,\varphi,\{1\},\{3\}\}$ **and** $m_Y=\{Y,\{a\},\{a,b\},\{c,d\},\{b,c,d\},\{a,c,d\}\}\cup\varphi$ **.**

Define $F:(X,m_X)\to(Y,m_Y)$ **by** $F(1)=a$ **,** $F(2)=\{b,c\}$**,** $F(3)=d$ **, then** F **is l.s.**θM **-continuous , but not lower** $M-$ **continuous.**

Theorem.5.19. For a multifunction $F:(X,m_X)\to(Y,m_Y)$ **, the following properties are equivalent:**

i- F **is l.s.**θM **-continuous.**

ii- $F^{-}(V)=\theta M-\text{int}(F^{-}(V))$ **,** $\forall V\in m_Y$ **.**

iii- $F^{+}(K)=\theta M-cl(F^{+}(K))$ **,** $\forall K$ **is** m_Y **-closed.**

Proof: the proof is similar to that of theorem 5.7.

Theorem.5.20. For a l.s.θM **-continuous multifunction** $F:(X,m_X)\to(Y,m_Y)$ **, with** m_Y **satisfies** (β) **properties , then the**

following are hold:

i- $m\ cl(F(B))\ F(m\ cl(B))\ {}_{X\ Y}\theta - \subseteq \theta -$ **,** $\forall B \subseteq Y$

ii- $F(m\ cl(A))\ m\ cl(F(A))\ {}_{X\ Y}\theta - \subseteq \theta -$ **,** $\forall A \subseteq X$ **.**

iii- $F(m\ \mathrm{int}(B))\ m\ \mathrm{int}(F(B))\ {}_{Y\ X}$

$- \theta - \subseteq \theta - -$ **,** $\forall B \subseteq Y$ **.**

Proof: (i) Similar to the proof of theorem 5.8

(ii) Let $A \subseteq X$ **, since** $A \subseteq F\ (F(A))$ **,then**

$(\ )\ (\ (\ (\ ))\ (\ (\ (\ ))$

(1)

$m\ cl\ A\ m\ cl\ F\ F\ A\ F\ m\ cl\ F\ A\ {}_{Y}$

by

${}_{X\ X}\theta - \subseteq \theta - \subseteq \theta -$ **,then**

$F(m\ cl(A))\ m\ cl(F(A)\ {}_{X\ Y}\theta - \subseteq \theta -$ **.**

(iii)By

(ii) $F(m - cl(F\ (Y \setminus B)) \subseteq {}_{X}\theta\ m\ cl(F(F(Y \setminus B))\ m\ cl(Y \setminus B)\ {}_{X\ Y}\theta - \subseteq \theta -$ **and**

$F(m - cl(F\ (Y \setminus B)))\ {}_{X}\theta\ F(m\ cl(X \setminus F(B)))\ {}_{X}$

$\theta - - = F(X \setminus m\ \mathrm{int}(F(B)))\ {}_{X}$

$\theta - -$ **,the**

n

$F(X \setminus m\ \mathrm{int}(F(B)))\ {}_{X}$

$\theta - -\ m\ cl(Y \setminus B)\ {}_{Y} \subseteq \theta -$ **.Now** $X \setminus m\ \mathrm{int}(F(B))\ {}_{X}$

$\theta - - \subseteq$

$F\ (m - cl(Y \setminus B))\ {}_{Y}\theta\ F\ (Y \setminus m - \mathrm{int}(B))\ {}_{X}\theta\ X \setminus F(m\ \mathrm{int}(B))\ {}_{Y} - \theta -$ **, therefore**

$F(m\ \mathrm{int}(B))\ m\ \mathrm{int}(F(B))\ {}_{Y\ X}$

$- \theta - \subseteq \theta - -$ **.**

Remark.5.21.The converse of theorem (5.20) is not true in general to show that let us see the following example.

Example.5.22.

Let $X\ \ \{1,2.3,4\}$ **and** $Y\ \ \{a,b,c\}$ **with** $m\ \{X, \varphi, \{2\},\{4\}\{1,2,3\},\{1,3,4\}\}\ {}_{X}$ **and**

$m\ \{Y, ,\{a\},\{a,b\},\{a,c\}\}\ {}_{Y}\ \ \varphi$ **, define** $F : X \to Y$ **by** $F(1)\ \ \{a\}, F(2)\ \ \{b,c\}$**,**

$F(3)\ \ \{c\}, F(4)\ \ \{a\}$ **,then** $m\ cl(F(B))\ F(m\ cl(B))\ {}_{X\ Y}\theta - \subseteq \theta -$ **,** $\forall B \subseteq Y$ **,**

$F(m\ cl(A))\ m\ cl(F(A))\ {}_{X\ Y}\theta - \subseteq \theta -$ **,** $\forall A \subseteq X$ **and**

$F(m\ \mathrm{int}(B))\ m\ \mathrm{int}(F(B))\ {}_{Y\ X}$

$- \theta - \subseteq \theta - -$ **,** $\forall B \subseteq Y$ **but F is not l.s.**θM **-**

continuous, since $1 \in X$ **and**

$(1)(\ \ \{\ \})\ \{\ \}(\in)\ \neq \varphi\ {}_{Y}F\ a\ \mathrm{l}\ a\ m$ **,but** $1 \in \{1,2,3\},\{1,3,4\}$ **and** X

Then $2\in\{1,2,3\}$ **and** X **and** $F(m-cl(2))(\ \{b,c\})\ \{a\}\ \ \varphi\ _X l$ **,and** $4\in\{1,3,4\}$

$F(m-cl(4))(\ \{c\})\ \{a\}\ \ \varphi\ _X l$ **, hence for any** $_X U\in m$ **containing** 1

$\exists u\in U$ **s.t.** $F\ m-cl\ u\ V\ \ \varphi\ (\ _X(\ ))\ l$ **.**

Remark.5.23. Every lower$\theta M-$ **continuous is lower**$\theta\ _*M-$ **continuous but the converse is not true in general and since every** $-\ _Y\theta m$ **open set contain** $-\ _Y m$ **open then every lower** $M-$ **continuous is** $\theta M-$ **continuous also but the converse is not true in general to show we give following examples.**

Examples.5.24. (i)Let $X\ \ \{1,2,3\}$**,**$m\ \{X,\varphi\ ,\{1\},\{3\}\}\ _X$ **and** $(\ ,\ )\ _Y Y\ m$ **as in example (5.12(i)),define** $F:X\to Y$ **by**

$F(1)\ \ \{a\}$ **,** $F(2)\ \ \{b,c\},\ F(3)\ \ \{d\}$ **,then** F **is lower**$\theta\ _*M-$**continuous but not lower**$\theta M-$ **continuous.**

(ii)In example (5.21) F **is lower**$\theta M-$ **continuous but not lower** $M-$ **continuous.**

Theorem.5.25.For a multifunction $F:(\ ,\ )\ _X X\ m\to(\ ,\ )\ _Y Y\ m$ **,the following properties are equivalent:-**

i- F **is lower**$\theta\ _*M-$ **continuous.**

ii- $F^-(V)=m\ \mathrm{int}(F(V))\ _X$

$\theta-{}_-$ **for every** $_Y V\in\theta m$ **.**

iii- $F\ \ (K)=m\ cl(F(K))\ _X$

$\theta-$ **for every** $-\ _Y\theta m$ **closed set** K **.**

iv- $m-cl(F\ \ (B))\subseteq\ _X\theta\ F(\ m\ cl(B))\ _Y\ \ \theta-$ **for every subset** $B\in\ _Y m$ **.**

v- $F(\ m\ cl(A))\ m\ cl(F(A))\ _X\ _Y\theta-\subseteq\theta-$ **, for every subset** $_X A\in m$ **.**

vi- $F^-(\ m-\mathrm{int}(B))\subseteq\ _Y\theta\ m\ \mathrm{int}(F(B))\ _X$

$\theta-{}_-$ **for every subset** $B\in\ _Y m$ **.**

Proof:-the proof is similar to that of theorem (5.18 and 5.19).

REFERENCES

1- Noiri,T.and Popa,V.;"On upper and lower M-continuous multifunction ",Filomat 14,pp.(73-86),(2000)

2- M.Caldas,S.Jafari and M.M.Kovar"Some Properties of θ −open sets ;Divulgaciones Matematicas,Vol.12,N0.2,pp(161-169), (2004)

3- Willard,S;"General topology"Addison-Wesley,Inc.Mass.(1970).