On δ-small M-Projective Modules

Nuhad S. Al-Mothafar, Munther T. Mohammed*
Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Abstract
In this paper we study the concepts of δ-small M-projective module and δ-small M-pseudo projective Modules as a generalization of M-projective module and M-Pseudo Projective respectively and give some results.

Keywords: δ-small M-projective modules, δ-small pseudo projective, δ-small M-pseudo projective modules.

حول مقاس M الاسقاطي من النوع δ الصغير

نهاد سالم المظفر، منذر طاهر محمد*
قسم الرياضيات، كلية العلوم، جامعة بغداد، بغداد، العراق .

الخلاصة
في هذا البحث ندرس مفهوم مقاس M الاسقاطي من النوع δ الصغير كتعميم لمقاس M الاسقاطي وكذلك ندرس مفهوم مقاس M الاسقاطي الزائف من النوع δ الصغير كتعميم لمقاس M الاسقاطي الزائف.

1. Introduction

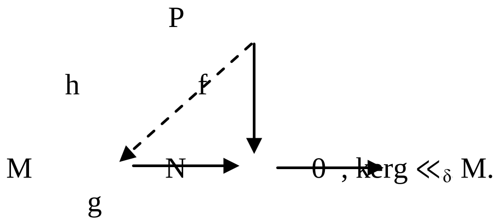
All rings in this paper are associative rings with identity, and all modules are unitary left R-modules. Let M be an R-module. A submodule A of M is called essential (denoted by $A \subseteq_e M$) if every nonzero submodule of M has a nonzero intersection with A [1]. If A is a submodule of M, then the annihilator of A (denoted by Ann (A)) is defined as Ann (A) = $\{r \in R \mid rA = 0\}$ [1]. If M is R-module, then $Z(M) = \{x \in M : Ann(A) \subseteq_e R\}$ is called the singular submodule of M . If Z(M) = M ,then M is called the singular module. If Z(M) = 0 then M is called nonsingular module [1]. A submodule N of a module M is called δ-small in M (denoted by $N \ll_\delta M$), if whenever N + X = M with M/X singular, we have X = M [2]. An epimorphism f: M → N is said to split if there exists a homomorphism g: N → M with fog = I_N [3]. A non-zero module M is called δ-hollow, if every proper submodule in M is δ-small in M [4]. An R-module P is called M-projective, if for any epimorphism g: M → N and any homomorphism f: P → N, there exists a homomorphism h: P → M such that goh = f [5]. A module P is called projective if it is M-projective for every R-module M [3]. Let N and L be submodules of M. N is called a δ-supplement of L if M = N + L and $N \cap L \ll_\delta N$ [4]. A module M is called Semisimple if it is a direct sum of simple modules [3]. An epimorphism g: M → N is said to be δ-small epimorphism if ker g is δ-small in M. [6]. A homomorphism f: M→N is said to be factor through g and h. if it is the composite of homomorphisms $f = g \circ h$ [7]. A module N is called M-pseudo projective if for every submodule A of M, any epimorphism α: N → M/A can be lifted to a homomorphism β: N → M [8].

*Email: munthertaher2@yahoo.com

2. δ-small M-Projective Modules

In this section, we introduce the definition of δ-small M-Projective Modules as a generalization of M-projective module. Also we introduce the definition of δ-small short exact sequence.

Definition (2.1): let N and M be modules. Then N is called δ-small M-projective, if for any given module A, any δ-small epimorphism g: M → A and any homomorphism f: N → A, there exists a homomorphism h: N → M such that goh = f. i.e. the following diagram is commutative:



Definition (2.2): A module N is called δ-small projective if it is δ-small M-projective for every R-module M,[6].

Definition (2.3): Let K, M, N be modules A short exact sequence

$0 \to K \xrightarrow{f} M \xrightarrow{g} N \to 0$ is said to be δ-small short exact sequence if $Kerg \ll_\delta M$.

Proposition (2.4): Let U and M be modules, the following are equivalent:

(a) U is a δ-small M-projective module;

(b) For every δ-small short exact sequence with middle term

$0 \to K \to M \to N \to 0$, the sequence

$0 \to Hom(U,K) \xrightarrow{Hom(I,f)} Hom(U,M) \xrightarrow{Hom(I,g)} Hom(U,N) \to 0$ is short exact;

(c) For any δ-small submodule K of M, any homomorphism

h: U → M/K factor through the natural epimorphism π: M → M/K.

Proof: (a ⇒ b) By proposition (16.6 in [7]) (b) holds. It is enough to show that, Hom(1, g) is an epimorphism. Let $f_1 \in Hom(U, N)$. Since g is a δ-small epimorphism and U is a δ-small M-projective module, there exists a homomorphism h: U → M such that goh = f_1.

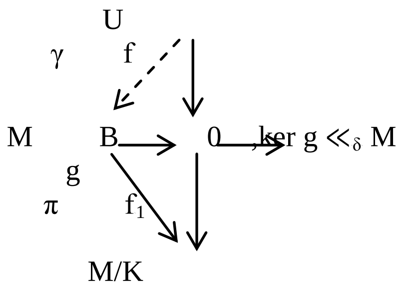
Now, Hom(1, g)(h) = goh = f_1.

(b ⇒ c) Let K be a δ-small submodule of M and let h: U → M/K be an epimorphism. Consider the following δ-small short exact sequence:

$0 \to K \xrightarrow{i} M \xrightarrow{\pi} N \to 0$ where i is the inclusion homomorphism and π is the natural epimorphism. By (b), the homomorphism Hom(I, π): Hom(U, M) → Hom(U, M/K) is an epimorphism.

i.e. there exists a homomorphism f ∈ Hom(U, M) such that h = Hom(I, π)(f) = πof.

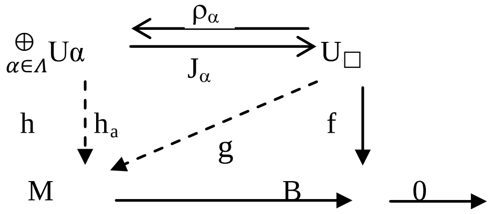
(c ⇒ a) Let g: M → B be a δ-small epimorphism and f: U → B be any homomorphism. Consider the following diagram:



Where K = Kerg, π: M → M/K is the natural epimorphism and f_1: B → M/K is an isomorphism. By (c), there exists a homomorphism γ: U → M such that πoγ = $f_1$of. and by (the factor theorem p.45 in [7]) we have $f_1$og = π. Now, $f_1$ogoγ = πoγ = $f_1$of. Thus goγ = f, since f_1 is an isomorphism.

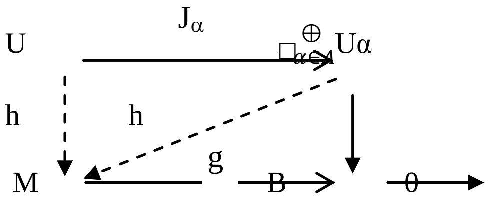
Proposition (2.5): Let M be an R-module and $\{U_\alpha\}_{\alpha\in\Lambda}$ be an indexed set of Modules. Then $\bigoplus_{\alpha\in\Lambda} U\alpha$ is a δ-small M-projective if and only if every U_α is a δ-small M-projective.

Proof: (⇒) Let $\underset{\alpha\in\Lambda}{\oplus}U\alpha$ be a δ-small M-projective and let $\alpha\in\Lambda$. Consider the following diagram:



Where g: M → B is a δ-small epimorphism, f: $U_\alpha \to B$ is any homomorphism, ρ_α and J_α are the projections and the injection homomorphisms, respectively. Since $\underset{\alpha\in\Lambda}{\oplus}U\alpha$ is δ-small M-projective, then there exists a homomorphism h: $\underset{\alpha\in\Lambda}{\oplus}U\alpha \to M$ such that $goh = fo\rho_\alpha$. Let $h_\alpha = hoJ_\alpha$: $U_\alpha \to M$. ⇒ $goh_\alpha = gohoJ_\alpha = fo\rho_\alpha oJ_\alpha = foI = f$.

(⇐) Let g : M → B be a δ-small epimorphism and let f : $\underset{\alpha\in\Lambda}{\oplus}U\alpha \to B$ be any homomorphism.For each $\alpha\in\Lambda$, consider the following diagram:



Where J_α: $U_\alpha \to \underset{\alpha\in\Lambda}{\oplus}U\alpha$ is the injection homomorphism. Since U_α is δ-small M-projective, for each $\alpha\in\Lambda$, so there exists h_α: $U_\alpha \to M$, such that $g_\circ h_\alpha = f_\circ J_\alpha$; for each $\alpha\in\Lambda$. Define h: $\underset{\alpha\in\Lambda}{\oplus}U\alpha \to M$ by $h(\psi) = \sum_{\alpha\in\Lambda} h_\alpha(\psi(\alpha))$. Clearly h is well-defined and a homomorphism.

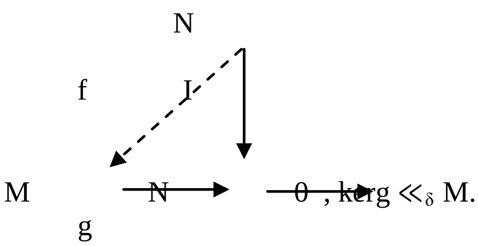
Now $(g_\circ h)(\psi) = g(h(\psi)) = g\left(\sum_{\alpha\in\Lambda} h_\alpha(\psi(\alpha))\right)$

$= \sum_{\alpha\in\Lambda}(g_\circ h_\alpha)(\psi(\alpha)) = \sum_{\alpha\in\Lambda}(f_\circ J_\alpha)(\psi(\alpha)) = f\left(\sum_{\alpha\in\Lambda} J_\alpha(\psi(\alpha))\right) = f(\psi)$.

Thus $\underset{\alpha\in\Lambda}{\oplus}U\alpha$ is δ-small M-projective module.

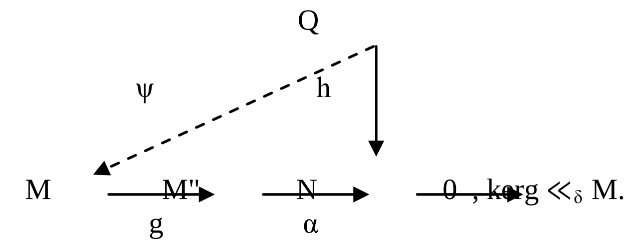
Proposition (2.6): If N is δ-small M-projective module, then every δ-small epimorphism g: M → N splits.

Proof: Let N be a δ-small M-projective, I: N → N be the identity and g: M → N be a δ-small epimorphism, then by δ-small M-projectivity there exists a homomorphism f: N → M such that g ∘ f = I. so the δ-small epimorphism g: N → M splits.



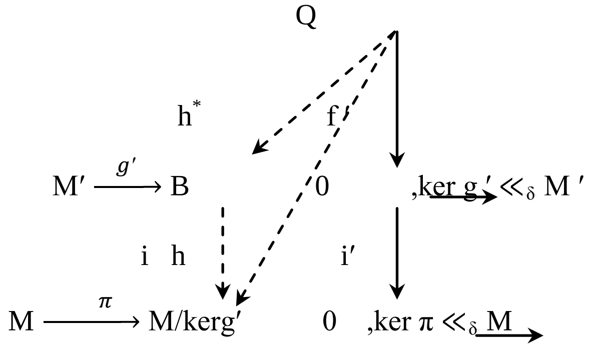
Proposition (2.7): Let M and Q be an R-module. If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is a δ-small short exact sequence and Q is a δ-small M-projective, then Q is δ-small M′ and M′′-projective.

Proof: First we show that Q is a δ-small M′′-projective, let α: M′′→ N be a δ-small epimorphism and let h: Q → N be any homomorphism. Consider the following diagram:



Since α and g are δ-small epimorphism so $\alpha \circ g$ is δ-small epimorphism [6], and since Q is δ-small M-projective, there exists a homomorphism ψ: $Q \to M$ such that $\alpha \circ g \circ \psi = h$, i.e., $g \circ \psi$ is the required homomorphism.

Now to show that Q is δ-small M'-projective, we may assume that $M' \leq M$, let g': $M' \to B$ be a δ-small epimorphism and f': $Q \to B$ be homomorphism. Consider the following diagram:



Where i be the inclusion homomorphism and π is the natural epimorphism.

Define i': $B \to M/Kerg'$ by $i'(b) = a + Kerg'$, where $b = g'(a)$, for some $a \in M'$. Its clear that g is well define and homomorphism. Since Q is a δ-small M-projective module, there exists a homomorphism h: $Q \to M$ such that $\pi \circ h = i' o f'$. We claim that $h(Q) \leq M'$. Let $w \in h(Q)$, then there exists $q \in Q$ such that $w = h(q)$. Now, $\pi h(q) = i' \circ f'(q) = i' \circ g'(a)$ for some $a \in M'$.

Hence $\pi h(q) = a + Kerg'$ and therefore $a - h(q) \in Kerg' \leq M'$. Thus $h(q) \in M'$ and consequently $h(Q) \leq M'$. Define h^*: $Q \to M'$ by $h(x) = h^*(x)$, for all $x \in Q$. Now, $i' \circ g' \circ h^* = \pi \circ i \circ h^* = \pi \circ h^* = \pi \circ h = i' \circ f'$.

Since i' is a monomorphism, we get $g' \circ h^* = f'$.

Hence Q is δ-small M'-projective module.

Corollary (2.8): Let Q be a δ-small M-projective module, if $N \subseteq M$, then Q is δ-small N-projective and δ-small M/N-projective.

Proof: Its clear from the short exact sequence $0 \to N \to M \to M/N \to 0$ and proposition (2.8).

Proposition (2.9): If M is δ-hollow module then every M-projective module is δ-Small M-projective.

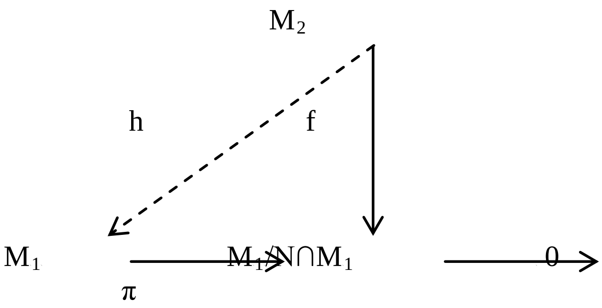
Proof: Follows by the fact every submodule of M is δ-small in M.

Proposition (2.10): Let M_1 and M_2 be modules, with $M = M_1 \oplus M_2$, then the following are equivalent:

(1) M_2 is a δ-small M_1-projective;

(2) For any submodule N of M, such that M_1 is a δ-supplement of N in M, there exists a submodule N_1 of N such that $M = M_1 \oplus N_1$.

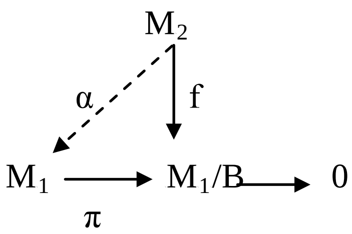
Proof: ($1 \Rightarrow 2$) Let M_1 be a δ-supplement of a submodule N in M, then $M = N + M_1$ with $N \cap M_1 \ll_\delta M_1$. Let π: $M_1 \to M_1/N \cap M_1$ be the natural epimorphism. Define f: $M_2 \to M_1/N \cap M_1$ by $f(x) = y + N \cap M_1$, for all $x \in M_2$, we have $x = y + n$, for some $y \in M_1$ and $n \in N$. Cleary f is well-defined and a homomorphism. Consider the following diagram:



Since M_2 is a δ-small M_1-projective, there exists a homomorphism h: $M_2 \to M_1$, such that $\pi \circ h = f$. Define $N_1 = \{y - h(y): y \in M_2\}$. We claim that $N_1 \subseteq N$. Let $x \in N_1$, then $x = w - h(w)$, for some $w \in M_2$. Now, $\pi h(w) = f(w)$. Since $M = N + M_1$ and $w \in M_2$, then $w = n + v$ for some $n \in N$ and $v \in M_1$. But $h(w)+N\cap M_1 = f(w) = v + N\cap M_1$. This implies that $h(w) - v \in N$ and thus $w - h(w) \in N$, i.e., $x \in N$. It is clear that $M = M_1 + N_1$. Let $w \in M_1 \cap N_1$, so $w = y - h(y)$ for some $y \in M_2$. Thus $w + h(y) = y = 0$. Thus $w = 0$. Hence $M = M_1 \oplus N_1$.

(2 ⇒ 1) Let π: $M_1 \to M_1/B$ be the natural epimorphism, where $B \ll_\delta M_1$ and let f: $M_2 \to M_1/B$, Define $N = \{x - y \mid f(x) = \pi(y)$, where $x \in M_2, y \in M_1\}$. It is clear that $M = M_1 + N$. We claim that $N \cap M_1 \subseteq B$. Let $w \in N \cap M_1$, so $w \in N$ and hence $w = m_2 - m_1$, for some $m_2 \in M_2$, $m_1 \in M_1$, where $f(m_2) = \pi(m_1)$. Thus $w + m_1 = m_2 = 0$, since $M = M_1 \oplus M_2$. Therefore $\pi(m_1) = 0$ which implies that $m_1 \in B$ and hence $w \in B$. But $B \ll_\delta M_1$, thus $N \cap M_1 \ll_\delta M_1$. Thus M_1 is a δ-supplement of N in M. By (2), there exists a submodule N_1 of N such that $M = M_1 \oplus N_1$. Define α: $M_2 \to M_1$ by $\alpha(w) = v$, where $w = n + v$ for some $n \in N_1$ and $v \in M_1$. Clearly α is well-defined and homomorphism.

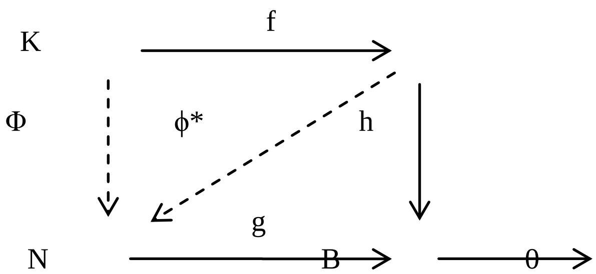
Now for the diagram



Let $w \in M_2$, then $w = n + v$, where $n \in N_1$ and $v \in M_1$, but $n \in N$, so $n = x - y$, where $f(x) = \pi(y)$. Hence $w = x - y + v$ which implies that $w - x = v - y \in M_1 \cap M_2 = 0$. Thus $w = x$ and $v = y$. Therefore $\pi\alpha(w) = \pi(v) = \pi(y) = f(x) = f(w)$. Consequently M_2 is a δ-small M_1-projective module.

Proposition (2.11): Let M, N and K be modules, where K is δ-small projective. Let f: $K \to M$ be an epimorphism. Then M is δ-small N-projective if for every homomorphism ϕ: $K \to N$, there exists a homomorphism ϕ*: $M \to N$ such that $\phi^* o f = \phi$.

Proof: Let g: $N \to B$ be δ-small epimorphism and h: $M \to B$ be any homomorphism. Consider the following diagram:

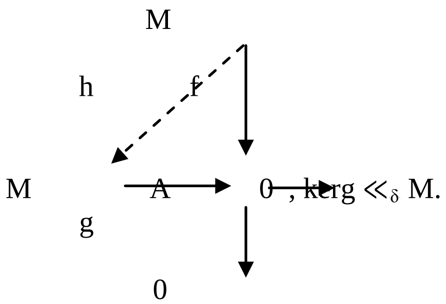


By δ-small projectivity of K, there exists a homomorphism ϕ: K → N, such that goϕ = hof. By our hypothesis, there exists a homomorphism ϕ^*: M → N, such that $\phi^*$of = ϕ, and so go$\phi^*$of = goϕ = hof. Now For m ∈ M, we have (goϕ^*)(m) = g(ϕ^*(m)) = g(ϕ^*(f(x))), where m = f(x), for some x ∈ K. Hence (goϕ^*)(m) = (go$\phi^*$of)(x) = (goϕ^*)(f(x)) = (goϕ)(x) = h(f(x)) = h(m) ⇒ goϕ^* = h. Therefore M is δ-small N-projective module.

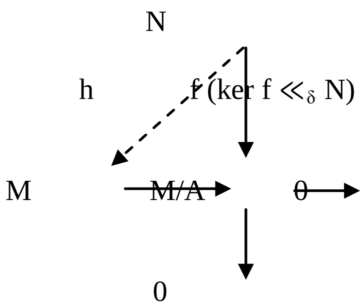
3. δ-Small M-Pseudo Projective Modules

In this section, we give new definitions, definitions of δ-Small pseudo projective module and δ-Small M-Pseudo Projective Module as a generalization of pseudo projective module and M-Pseudo Projective Module respectively and give some results. Recall that An R-module M is called pseudo projective if for any given module A and epimorphisms f: M→A and g: M→ A, there exists an h in End (M) such that f = g o h. also recall that a module N is called M-pseudo projective if for every submodule A of M, any epimorphism α: N → M/A can be lifted to a homomorphism β: N → M.

Definition (3.1): An R-module M is said to be δ-small pseudo projective if for any module A, with δ-small epimorphism g: M → A and epimorphism f: M → A there exists an h ∈ End (M) such that f = g ∘ h. i.e. the following diagram is commutative:

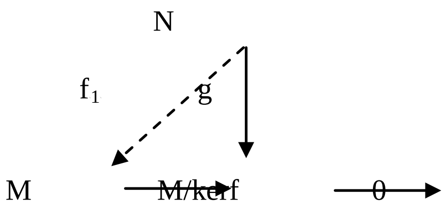


Definition (3.2): An R-module N is called δ-small M-pseudo projective module if for any submodule A of M, any δ-small epimorphism f: N → M/A can be lifted to a homomorphism h: N → M. i.e. the following diagram is commutative:



Proposition (3.3): Let N be a δ-small M-pseudo projective module, then any epimorphism f: M → N splits.

Proof: Let f: M → N be an epimorphism. Then N ≅ M/ker(f) so g: N → M/kerf is an isomorphism, since N is δ-small M-pseudo projective then g can be lifted to homomorphism f_1: N → M. thus f ∘ f_1 is the identity map, therefore the epimorphism f: M → N splits.

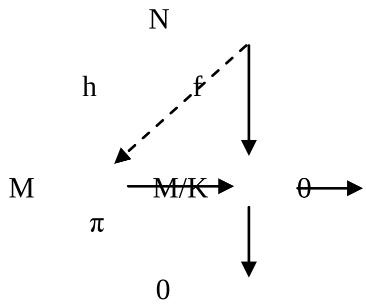


Proposition (3.4): Let N be a δ-hollow R-module the following conditions are equivalent:

(1) N is δ-small M-pseudo projective module.

(2) N is M-pseudo projective module.

Proof(1) ⇒ (2) Let N be a δ-small M-pseudo projective module. Let K be any submodule of M, let f: N → M/K any epimorphism. Since N is δ-hollow module so every proper submodules of N are δ-small in N. so Ker f $\ll_\delta$ N, and by (1) the homomorphism f can be lifted to a homomorphism h: N → M. such that πog = f. with π: M → M/K. i.e. the following diagram is commutative:

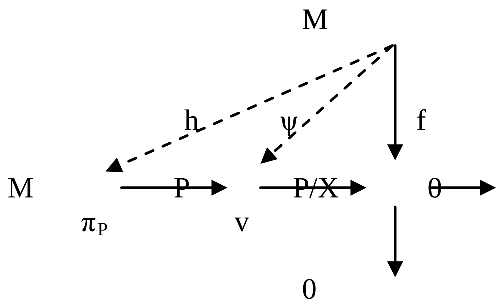


Thus, N is M-pseudo projective module.

(2) ⇒ (1), it clear by definition.

Proposition (3.5): If M = P ⊕ N is δ-small pseudo projective then P ⊕ N is δ-small P-pseudo projective as well as δ-small N-pseudo projective.

Proof: Let f: M → P/X be any δ-small epimorphism where X is a submodule of P, π_P: M → P be the projection map and ν: P → P/X be the natural epimorphism. Then by δ-small pseudo projectivity of M there exists h: M →M such that the following diagram is commutative:



i.e. f = v∘π_P∘h, define ψ = π_P∘h thus v∘ψ = f and hence M is δ-small P-pseudo-projective. Similarly we can show that M is δ-small N-pseudo-projective.

References :

1. Goodearl K. R. **1976**. *Ring theory, Non-Singular Rings and Modules,* Mercel Dekker, New York. pp. 15-40.
2. Zhou Y. **2000**. Generalization of Perfect, Semiperfect and Semiregular Rings, Algebra colloquium, 7(3). pp:305-318.
3. Wisbauer R. **1991**. Foundations of Modules and Rings theory, Gordon and Breach, Philadelphia. pp:57-166
4. Nematollahi M. J. **2009**, On δ-supplemented modules, Tarbiat Moallem University, 20th Seminar on Algebra, 2-3 Ordibehesht,. pp:155-158.
5. Azumayya G. Mbuntum. F and Varadarajan. K. **1957**. On M-projective and M-injective modules, *pacific journal of mathematics*, 59(1), pp:9-16.
6. Almothafar N. S. and Yassin, S. M, **2013** On δ-small projective module, *Iraqi Journal of science*, 54, pp:855-860.
7. Anderson, F. W. and Fuller K. R. **1974**. *Rings and Categories of Modules*, Sipringer-Verlag, New York. pp:45-185.
8. Talebi Y. and Gorji I. K. **2008**. On Pseudo-Projective and Pseudo Small Projective Modules,

International Journal of Algebra, 2(10), pp:463-468.